

# MANIFOLDS, STRUCTURES CATEGORICALLY

G.V. KONDRATIEV

**ABSTRACT.** A notion of general manifolds is introduced. It covers all usual manifolds in mathematics. Essentially, it is a way how to get a bigger 'fibration' over a site which locally coincides with a given one. An enrichment with generalized elements is regarded which allows to see hom-sets of a given category as (almost) objects and to transfer some technics from objects onto hom-sets. Lifting problem for a group action and actions of group objects are also included.

## 1. Fibrations and cofibrations

(Co)fibrations play role of structures over objects in a given category which can be transported along morphisms. Transport system is called (co)cartesian morphisms. The situation is very similar to fibrations with connection as in Differential Geometry.

**Definition 1.1.** For a functor  $p : \mathbf{E} \rightarrow \mathbf{B}$

- morphism  $f : B' \rightarrow p(E)$  has a **cartesian lifting**  $\tilde{f} : E' \rightarrow E \in Ar \mathbf{E}$  if  $\forall f' : E'' \rightarrow E$  such that  $p(f')$  factors through  $f$  (i.e.,  $\exists g : p(E'') \rightarrow B'$  such that  $p(f') = f \circ g$ )  $f'$  itself uniquely factors through  $\tilde{f}$  over the base factorization (i.e.,  $\exists \tilde{g} : E'' \rightarrow E'$  such that  $f' = \tilde{f} \circ \tilde{g}$  and  $p(\tilde{g}) = g$ ) [Jac]

$$\begin{array}{ccccc}
 E'' & & & & \\
 \swarrow \exists! \tilde{g} & & \searrow \forall f' & & \\
 p(E'') & & E' & \xrightarrow{\tilde{f}} & E \\
 \swarrow \forall g & & \searrow p(f') & & \\
 B' & \xrightarrow{f} & p(E) & & 
 \end{array}$$

- morphism  $f : p(E) \rightarrow B'$  has a **cocartesian lifting**  $\tilde{f} : E \rightarrow E' \in Ar \mathbf{E}$  if  $\forall f' : E \rightarrow E''$  such that  $p(f')$  factors through  $f$  (i.e.,  $\exists g : B' \rightarrow p(E'')$  such that  $p(f') = g \circ f$ )  $f'$  itself uniquely factors through  $\tilde{f}$  over the base factorization (i.e.,  $\exists \tilde{g} : E' \rightarrow E''$  such that  $f' = \tilde{g} \circ \tilde{f}$  and

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$$p(\tilde{g}) = g) \text{ [Jac]}$$

$$\begin{array}{ccccc}
 & & & & E'' \\
 & & \nearrow \forall f' & \nearrow \exists! \tilde{g} & \\
 E & \xrightarrow{\tilde{f}} & E' & & p(E'') \\
 & \nearrow p(f') & \nearrow \forall g & & \\
 p(E) & \xrightarrow{f} & B' & & 
 \end{array}$$

□

**Remark.** (Co)cartesian morphisms if exist are unique up to vertical isomorphism ( $v : E \rightarrow E' \in Ar \mathbf{E}$  is **vertical** if  $p(v) = 1_B$  for some  $B \in Ob \mathbf{B}$ ).

**Definition 2.1.2.** A functor  $\begin{array}{c} \mathbf{E} \\ p \downarrow \\ \mathbf{B} \end{array}$  is called [Jac, Str]

- **fibration** if for each  $f : B' \rightarrow p(E) \in Ar \mathbf{B}$  there exists cartesian lifting  $\tilde{f} : E' \rightarrow E \in Ar \mathbf{E}$
- **cofibration** if for each  $f : p(E) \rightarrow B' \in Ar \mathbf{B}$  there exists cocartesian lifting  $\tilde{f} : E \rightarrow E' \in Ar \mathbf{E}$
- **bifibration** if it is both fibration and cofibration

□

Subcategory  $\mathbf{E}_B := p^{-1}(B, 1_B) \hookrightarrow \mathbf{E}$  is called **fiber** over  $B$ .  $\mathbf{E}_B$  consists of all vertical morphisms over  $B$ .

### Examples

1. For a category  $\mathbf{C}$  with pullbacks **codomain fibration** is  $\begin{array}{c} \mathbf{C} \rightarrow \\ \text{cod} \downarrow \\ \mathbf{C} \end{array}$ ,  $\text{cod} \left( \begin{array}{ccc} \bullet & \xrightarrow{\Phi} & \bullet \\ \downarrow & & \downarrow \\ \bullet & \xrightarrow{f} & \bullet \end{array} \right) = f$ ,

cartesian lifting is a pullback square

2. For a category  $\mathbf{C}'$  with pushouts **domain cofibration** is  $\begin{array}{c} \mathbf{C}' \rightarrow \\ \text{dom} \downarrow \\ \mathbf{C}' \end{array}$ ,  $\text{dom} \left( \begin{array}{ccc} \bullet & \xrightarrow{\Phi} & \bullet \\ \downarrow & & \downarrow \\ \bullet & \xrightarrow{f} & \bullet \end{array} \right) = \Phi$ ,

cocartesian lifting is a pushout square

3. [Kom] Denote **Rng-Mod**, category of left modules with variable rings.  $Ob(\mathbf{Rng-Mod})$  are pairs  $(R, M)$  where:  $R$  is a ring,  $M$  is an  $R$ -module.  $Ar(\mathbf{Rng-Mod})$  are pairs  $(\varphi : R_1 \rightarrow R_2, f : M_1 \rightarrow M_2)$  such that  $\forall r \in R_1, m \in M_1 \quad f(r \cdot m) = \varphi(r) \cdot f(m)$ . Then 'projection on first component'  $\mathbf{Rng-Mod} \xrightarrow{p_1} \mathbf{Rng}$  is a bifibration. If  $\varphi : R_1 \rightarrow R_2 \in Ar \mathbf{Rng}$  then **direct image** of  $(R_1, M_1)$  over  $\varphi$  is  $(R_2, R_2 \otimes_{R_1} M_1)$ . Cocartesian lifting of  $\varphi$  is  $(\varphi, \bar{\varphi})$  with  $\bar{\varphi} : M_1 \rightarrow R_2 \otimes_{R_1} M_1 : m \mapsto 1 \otimes m$ .

$$\begin{array}{ccc}
 & & (R_3, M_3) \\
 & \nearrow^{(\psi \circ \varphi, f)} & \\
 (R_1, M_1) & \xrightarrow{(\varphi, \bar{\varphi})} & (R_2, R_2 \otimes_{R_1} M_1) \\
 & \searrow_{(\psi, \psi \cdot f)} &
 \end{array}$$

$$\begin{array}{ccccc}
 & & & & R_3 \\
 & & \nearrow^{\psi \circ \varphi} & & \\
 R_1 & \xrightarrow{\varphi} & R_2 & \xrightarrow{\psi} & R_3
 \end{array}$$

where:  $\psi \cdot f := \mu \circ (\psi \otimes f) : R_2 \otimes_{R_1} M_1 \xrightarrow{\psi \otimes f} R_3 \otimes_{R_1} M_3 \xrightarrow{\mu} M_3$ ,  $\mu$  is module multiplication,  $\psi(r_1 \cdot r_2) \cdot f(m) = (\psi \cdot f)(r_1 \cdot r_2 \otimes m) = \psi(r_1) \cdot (\psi \cdot f)(r_2 \otimes m) = \psi(r_1) \cdot (\psi(r_2) \cdot f(m))$ .

**Inverse image** of  $(R_1, M_1)$  over  $\alpha : R \rightarrow R_1$  is  $(R, M_1)$  with action  $(r, m) \mapsto \alpha(r) \cdot m$ .  
**Cartesian lifting** of  $\alpha$  is  $(\alpha, 1_{M_1})$ .

$$\begin{array}{ccc}
 (R', M') & & \\
 \searrow^{(\alpha \circ \beta, f)} & & \\
 & \searrow_{(\beta, f)} & \\
 & (R, M_1) & \xrightarrow{(\alpha, 1_{M_1})} (R_1, M_1)
 \end{array}$$

$$\begin{array}{ccccc}
 R' & & & & \\
 \searrow^{\alpha \circ \beta} & & & & \\
 & \searrow_{\beta} & & & \\
 & R & \xrightarrow{\alpha} & R_1
 \end{array}$$

#### 4. (Differential Equations)

$$\mathbf{Set} \xleftarrow{p_1} \mathbf{Top} \xleftarrow{p_2} \mathbf{Diff} \xleftarrow{p_3} \mathbf{Diff}^{\rightarrow} \xleftarrow{p_4} \mathbf{Jet}^{\infty}(\mathbf{Diff}^{\rightarrow}) \xleftarrow{p_5} \mathbf{Sub}(\mathbf{Jet}^{\infty}(\mathbf{Diff}^{\rightarrow}))$$

where  $\mathbf{Diff}^{\rightarrow}$  is a full subcategory of  $\mathbf{Diff}^{\rightarrow}$  consisting of surjective submersions (surjective maps with surjective differential at each point),  $\mathbf{Jet}^{\infty}(\mathbf{Diff}^{\rightarrow})$  is the corresponding category of  $\infty$ -jet-bundles,  $\mathbf{Sub}(\mathbf{Jet}^{\infty}(\mathbf{Diff}^{\rightarrow}))$  is the category of subobjects of  $\infty$ -jet-bundles (differential relations),

$$T' \xrightarrow{v} T$$

$p_1 : \mathbf{Top} \rightarrow \mathbf{Set}$  is a fibration ( $\forall$  arrow  $S' \xrightarrow{u} p_1(T) \exists$  Cartesian 'completion' ,

$$S' \xrightarrow{u} p_1(T)$$

$T'$  has initial topology w.r.t.  $u$ ),

$p_2 : \mathbf{Diff} \rightarrow \mathbf{Top}$  is not a fibration if differentiable manifolds are regarded in usual sense (as locally Euclidian), but it is a fibration if differentiable manifolds are topological spaces endowed with a subsheaf of continuous functions closed under smooth operations,

$p_3 : \mathbf{Diff}^{\rightarrow} \rightarrow \mathbf{Diff}$  is a codomain fibration since pullback of a surjective submersion is a surjective submersion,

$p_4 : \mathbf{Jet}^{\infty}(\mathbf{Diff}^{\rightarrow}) \rightarrow \mathbf{Diff}^{\rightarrow}$  is not a fibration (if we admit arbitrary fibre bundle arrows between objects in  $\mathbf{Jet}^{\infty}(\mathbf{Diff}^{\rightarrow})$ ), but it is a structure over  $\mathbf{Diff}^{\rightarrow}$  (see 2.1.1),

$p_5 : \mathbf{Sub}(\mathbf{Jet}^\infty(\mathbf{Diff}^{\rightarrow})) \rightarrow \mathbf{Jet}^\infty(\mathbf{Diff}^{\rightarrow})$  is a 'subobject' fibration.

**Lemma 1.1.** *Every functor  $F : \mathbf{C} \rightarrow \mathbf{B}$  factors through a free fibration  $\mathbf{C} \xrightarrow{i} 1/F$  where*

$$\begin{array}{ccc} \mathbf{C} & \xrightarrow{i} & 1/F \\ & \searrow F & \downarrow \text{dom} \\ & & \mathbf{B} \end{array}$$

$$i : \mathbf{C} \rightarrow 1/F : \begin{cases} C \mapsto 1_{F(C)} & C \in \text{Ob } \mathbf{C} \\ (f : C \rightarrow C') \mapsto (F(f), f) & f \in \text{Ar } \mathbf{C} \end{cases}$$

□

Taking (co)cartesian morphism over  $f : B' \rightarrow B \in \text{Ar } \mathbf{B}$  depends on the choice of an object  $E \in \text{Ob } \mathbf{E}_B$  (respectively,  $E' \in \text{Ob } \mathbf{E}_{B'}$ ) and the choice of an arrow  $\tilde{f}_E : E' \rightarrow E$  (respectively,  $\tilde{f}_{E'} : E' \rightarrow E$ ) in the isomorphism class.

**Lemma 1.2.**

- For a (co)fibration  $\begin{array}{c} \mathbf{E} \\ \downarrow p \\ \mathbf{B} \end{array}$ , an arrow  $f : B' \rightarrow B \in \text{Ar } \mathbf{B}$ , and a choice  $\tilde{f}_E : E' \rightarrow E \ \forall E \in \text{Ob } \mathbf{E}_B$  (respectively,  $\tilde{f}_{E'} : E' \rightarrow E \ \forall E' \in \text{Ob } \mathbf{E}_{B'}$ ) there is a functor  $\mathbf{Cart}_f : \mathbf{E}_B \rightarrow \mathbf{E}_{B'}$  :

$$\left\{ \begin{array}{l} E \mapsto d(\tilde{f}_E) \\ (g : E_1 \rightarrow E) \mapsto \mathbf{Cart}_f(g) \end{array} \right. \quad \begin{array}{c} E \in \text{Ob } \mathbf{E}_B \\ \\ g \in \text{Ar } \mathbf{E}_B \end{array}$$

$$\text{ s.t. } \begin{array}{ccc} E'_1 & \xrightarrow{\tilde{f}_{E_1}} & E_1 \\ \downarrow \mathbf{Cart}_f(g) & & \downarrow g \\ E' & \xrightarrow{\tilde{f}_E} & E \end{array}$$

respectively,  $\mathbf{Cart}_f : \mathbf{E}_{B'} \rightarrow \mathbf{E}_B$  :

$$\left\{ \begin{array}{l} E' \mapsto c(\tilde{f}_{E'}) \\ (g : E'_1 \rightarrow E') \mapsto \mathbf{Cart}_f(g) \end{array} \right. \quad \begin{array}{c} E' \in \text{Ob } \mathbf{E}_{B'} \\ \\ g \in \text{Ar } \mathbf{E}_{B'} \end{array}$$

$$\text{ s.t. } \begin{array}{ccc} E'_1 & \xrightarrow{\tilde{f}_{E'_1}} & E_1 \\ \downarrow g & & \downarrow \exists! \mathbf{Cart}_f(g) \\ E' & \xrightarrow{\tilde{f}_{E'}} & E \end{array}$$

- For any two such choices of (co)cartesian morphisms  $\tilde{f}_E$ ,  $\forall E \in \mathbf{E}_B$  and  $\tilde{f}_{E'}$ ,  $\forall E' \in \mathbf{E}_{B'}$  the corresponding functors  $\mathbf{Cart}_f$  and  $\widetilde{\mathbf{Cart}}_f$  are isomorphic.

*Proof* is straightforward. See [Jac].

□

Another equivalent description of (co)fibrations is via (co)contravariant pseudofunctors  $\mathbf{B} \rightarrow \mathbf{CAT}$ .

**Proposition 1.1.**

- For each choice of (co)cartesian liftings for all arrows  $f : B' \rightarrow p(E)$  (or respectively,  $f : p(E) \rightarrow B'$ ) in the base category of a (co)fibration  $p \downarrow$  there is a corresponding pseudofunctor

$$F_p : \mathbf{B} \rightarrow \mathbf{CAT} : \begin{cases} B \mapsto \mathbf{E}_B & B \in \text{Ob } \mathbf{B} \\ f \mapsto \mathbf{Cart}_f & f \in \text{Ar } \mathbf{B} \end{cases}$$

- Conversely, for a given (co)contravariant pseudofunctor  $F : \mathbf{B} \rightarrow \mathbf{CAT}$  there is a (co)fibration

$$\begin{array}{c} \mathbf{E}_F \\ p_F \downarrow \\ \mathbf{B} \end{array} \quad (\text{Grothendieck construction}).$$

$$\begin{cases} \text{Ob } \mathbf{E}_F \text{ are pairs } \begin{pmatrix} E \\ B \end{pmatrix}, & E \in \text{Ob } F(B), \ B \in \text{Ob } \mathbf{B} \\ \text{Ar } \mathbf{E}_F \text{ are pairs } \begin{pmatrix} h \\ f \end{pmatrix} \in \mathbf{E}_F \left( \begin{pmatrix} E \\ B \end{pmatrix}, \begin{pmatrix} E' \\ B' \end{pmatrix} \right) & h \in F(B)(E, F(f)(E')), \ f \in \mathbf{B}(B, B') \end{cases}$$

$$1_{\begin{pmatrix} E \\ B \end{pmatrix}} := \begin{pmatrix} E \xrightarrow{\sim} F(1_B)(E) \\ 1_B \end{pmatrix},$$

$$\text{for } \begin{pmatrix} u \\ f \end{pmatrix} : \begin{pmatrix} E \\ B \end{pmatrix} \rightarrow \begin{pmatrix} E' \\ B' \end{pmatrix} \text{ and } \begin{pmatrix} v \\ g \end{pmatrix} : \begin{pmatrix} E' \\ B' \end{pmatrix} \rightarrow \begin{pmatrix} E'' \\ B'' \end{pmatrix} \text{ the composite}$$

$$\begin{pmatrix} v \\ g \end{pmatrix} \circ \begin{pmatrix} u \\ f \end{pmatrix} := \begin{pmatrix} w \\ g \circ f \end{pmatrix} \text{ where } w : E \xrightarrow{u} F(f)(E') \xrightarrow{F(f)(v)} F(f)(F(g)(E'')) \xrightarrow{\sim} F(g \circ f)(E'')$$

or respectively,

$$\begin{cases} \text{Ob } \mathbf{E}_F \text{ are pairs } \begin{pmatrix} E \\ B \end{pmatrix}, & E \in \text{Ob } F(B), \ B \in \text{Ob } \mathbf{B} \\ \text{Ar } \mathbf{E}_F \text{ are pairs } \begin{pmatrix} h \\ f \end{pmatrix} \in \mathbf{E}_F((E, B), (E', B')) & h \in F(B')(F(f)(E), E'), \ f \in \mathbf{B}(B, B') \end{cases}$$

$$1_{\begin{pmatrix} E \\ B \end{pmatrix}} := \begin{pmatrix} F(1_B)(E) \xrightarrow{\sim} E \\ 1_B \end{pmatrix},$$

$$\text{for } \begin{pmatrix} u \\ f \end{pmatrix} : \begin{pmatrix} E \\ B \end{pmatrix} \rightarrow \begin{pmatrix} E' \\ B' \end{pmatrix} \text{ and } \begin{pmatrix} v \\ g \end{pmatrix} : \begin{pmatrix} E' \\ B' \end{pmatrix} \rightarrow \begin{pmatrix} E'' \\ B'' \end{pmatrix} \text{ the composite}$$

$$\begin{pmatrix} v \\ g \end{pmatrix} \circ \begin{pmatrix} u \\ f \end{pmatrix} := \begin{pmatrix} w \\ g \circ f \end{pmatrix} \text{ where } w : F(g \circ f)(E) \xrightarrow{\sim} F(g)(F(f)(E)) \xrightarrow{F(g)(u)} F(g)(E') \xrightarrow{v} E'',$$

$p_F$  is the projection onto the bottom component in both cases.

- Two processes above are (weakly) inverse to each other.

*Proof* is straightforward. See [Jac].

$$\text{(It is essential for Grothendieck construction that every (co)fibration } p \downarrow \text{ is equivalent to } \begin{array}{c} \mathbf{E} \\ p \downarrow \\ \mathbf{B} \end{array})$$

(co)fibration  $p_2 \downarrow \mathbf{B}$  where  $\begin{cases} Ob p := \left\{ \begin{pmatrix} E \\ p(E) \end{pmatrix} \mid E \in Ob \mathbf{E} \right\} \\ Ar p := \left\{ \begin{pmatrix} f \\ p(f) \end{pmatrix} \mid f \in Ar \mathbf{E} \right\} \end{cases}$ ,  $p_2$  is the projection onto the bottom

component, and that every morphism in  $\mathbf{E}$  factors through (co)cartesian one)  $\square$

## 2. Almost structures

**Definition 2.1.** Structure of type  $\mathbf{E}$  on (objects of) category  $\mathbf{B}$  is a functor

$$\begin{array}{c} \mathbf{E} \\ \downarrow p \\ \mathbf{B} \end{array}$$

which is

- faithful
- admits lifting of iso's of type  $f : B' \xrightarrow{\sim} p(E)$  (or, the same,  $f : p(E) \xrightarrow{\sim} B'$ )
- each fiber  $\mathbf{E}_B$  is skeletal

$\square$

**Lemma 2.1.** Let  $p \downarrow \mathbf{B}$  be a structure on  $\mathbf{B}$ ,  $E', E'' \in Ob(\mathbf{E}_B)$  for some  $B \in Ob(\mathbf{B})$ .

The following are equivalent

- a)  $E' = E''$
- b)  $\forall E \in Ob \mathbf{E} \ p(\mathbf{E}(E, E')) = p(\mathbf{E}(E, E''))$
- c)  $\forall E \in Ob \mathbf{E} \ p(\mathbf{E}(E', E)) = p(\mathbf{E}(E'', E))$
- d)  $\forall E \in Ob \mathbf{E}_B \ \mathbf{E}_B(E, E') = \emptyset \iff \mathbf{E}_B(E, E'') = \emptyset$
- e)  $\forall E \in Ob \mathbf{E}_B \ \mathbf{E}_B(E', E) = \emptyset \iff \mathbf{E}_B(E'', E) = \emptyset$

*Proof.* a)  $\Rightarrow$  b), c), d), e) is obvious.

b)  $\Rightarrow$  a) Take  $E = E'$  and  $E = E''$  then  $\exists f : E' \rightarrow E''$  and  $\exists g : E'' \rightarrow E'$  such that  $p(f) = 1_B = p(g)$ . So,  $p(f \circ g) = p(g \circ f) = 1_B$ . By faithfulness of  $p$  and skeletal condition on  $\mathbf{E}_B$ ,  $f$  and  $g$  are trivial iso's.

c)  $\Rightarrow$  a) is the same as b)  $\Rightarrow$  a).

d)  $\Rightarrow$  a) Take  $E = E'$  and  $E = E''$  then  $\exists f : E' \rightarrow E''$ ,  $f$  is vertical, and  $\exists g : E'' \rightarrow E'$ ,  $g$  is vertical. So,  $f \circ g = 1_{E''}$ ,  $g \circ f = 1_{E'}$   $\Rightarrow E' \simeq E'' \Rightarrow E' = E''$ .

e)  $\Rightarrow$  a) is the same as d)  $\Rightarrow$  a).  $\square$

**Proposition 2.1.** For structure  $p \downarrow \mathbf{B}$  on  $\mathbf{B}$  each fiber  $\mathbf{E}_B$  is a poset category with (vertical)

cartesian morphisms, just identities.

*Proof.*  $\mathbf{E}_B$  is a preorder since  $\exists$  at most one morphism between objects ( $p$  is faithful).  $\mathbf{E}_B$  is a partial order since all isomorphic objects are the same (skeletal condition). Cartesian lift of iso is iso, so cartesian lift  $\widetilde{1_B}$  is an identity (since  $\mathbf{E}_B$  is a partial order).  $\square$

**Proposition 2.2.**

- Pullback of a fibration is a fibration,

$$\begin{array}{ccc} F^*\mathbf{E} & \xrightarrow{\bar{F}} & \mathbf{E} \\ p' \downarrow & & \downarrow p \\ \mathbf{A} & \xrightarrow{F} & \mathbf{B} \end{array}$$

- Pullback of a structure (on  $\mathbf{B}$ ) is a structure (on  $\mathbf{A}$ )

*Proof.*

- Pullbacks in  $\mathbf{CAT}$  exist and they are certain subcategories of direct products. Cartesian morphisms are preserved under pullbacks which is seen from the following diagram

$$\begin{array}{ccc} (A'', E'') & & E'' \\ \swarrow (u \circ w, \alpha) & & \searrow \alpha \\ (A', E') & \xrightarrow{(u, v)} & (A, E) \\ \swarrow (w, \beta) & & \searrow \beta \\ & & E' \xrightarrow{v} E \end{array}$$

$$\begin{array}{ccc} A'' & & FA'' \\ \searrow \forall w & & \searrow Fw \\ A' & \xrightarrow{u} & A \\ & & FA' \xrightarrow{Fu} FA \end{array}$$

(where:  $p(E) = F(A)$ ,  $p(\alpha) = F(u \circ w)$ ,  $v$  is cartesian over  $F(u)$ ).

- If  $\begin{array}{c} \mathbf{E} \\ p \downarrow \\ \mathbf{B} \end{array}$  admits lifting of iso's of type  $B' \xrightarrow{\sim} p(E)$  then  $\begin{array}{c} F^*\mathbf{E} \\ p' \downarrow \\ \mathbf{A} \end{array}$  admits lifting of iso's of the same type (obvious). If  $p$  is faithful then  $p'$  is faithful (assume,  $p'(u, v) = p'(u, v_1) = u$  then  $F(u) =$

$p(v) = p(v_1)$ , so,  $v = v_1$ ). Fibres of  $\begin{array}{c} F^*\mathbf{E} \\ p' \downarrow \\ \mathbf{A} \end{array}$  are skeletal (assume,  $(1_A, v) : (A, E') \xrightarrow{\sim} (A, E)$  is an iso in  $(F^*\mathbf{E})_A$  then  $v : E' \xrightarrow{\sim} E$  is an iso in  $\mathbf{E}_{F(A)}$ , so,  $E' = E$ , and  $(A, E') = (A, E)$ ).  $\square$

Partial cases of a pullback are 'fiber' and 'intersection of structures':

$$\begin{array}{ccc} \mathbf{E}_B & \longrightarrow & \mathbf{E} \\ \downarrow & & \downarrow p \\ 1 & \xrightarrow{B} & \mathbf{B} \end{array} \quad \begin{array}{ccc} \mathbf{E}_1 \wedge \mathbf{E} & \longrightarrow & \mathbf{E} \\ \downarrow & \searrow \pi & \downarrow p \\ \mathbf{E}_1 & \xrightarrow{p_1} & \mathbf{B} \end{array}$$

**Remark.** The notion of 'structure on objects' of a category was introduced in [Kom] in order to deal with usual structures in Differential Geometry like smooth structures or fibre bundles on topological spaces. However, it turned out too weak (no inverse and direct images) and too strict (skeletal fibres) simultaneously. Appropriate framework was created with theory of (co)fibrations. Nevertheless, a weaker notion of almost structure emphasizes direct connection with the main structure, which is the case of importance especially when almost structure is introduced on hom-sets.

**Proposition 2.3.** For each structure  $p \downarrow \begin{smallmatrix} \mathbf{E} \\ \mathbf{B} \end{smallmatrix}$  of type  $\mathbf{E}$  on objects of category  $\mathbf{B}$

- there is an embedding  $i_p : p \downarrow \begin{smallmatrix} \mathbf{E} \\ \mathbf{B} \end{smallmatrix} \hookrightarrow \mathbf{Set}^{\mathbf{E}^{op}} : \begin{cases} \left( \begin{smallmatrix} E \\ p(E) \end{smallmatrix} \right) \mapsto p(\mathbf{E}(-, E)) & \text{on objects} \\ \left( \begin{smallmatrix} v \\ p(v) \end{smallmatrix} \right) \mapsto p(\mathbf{E}(-, v)) & \text{on arrows} \end{cases}$
- $p(\mathbf{E}(-, E)) \hookrightarrow \mathbf{B}(p(-), p(E)) : \mathbf{E}^{op} \rightarrow \mathbf{Set}$  (hom-subfunctor)

*Proof.* Functoriality is obvious. Injectivity follows from Lemma 2.1.1.1.  $\square$

**Remark.** It means that every structure  $\mathbf{E}$  on objects in  $\mathbf{B}$  is **faithfully** representable by a specific subcategory of  $\mathbf{B}$ -hom-subfunctors (in which sufficient to take arrows of only simple type  $f \circ -$ ).

The reasonable question is if can an object  $E \in Ob \mathbf{E}$  be recovered from a functor  $F \hookrightarrow \mathbf{B}(p(-), p(E))$ ? If can it is unique but the answer is no, in general. Even when it is impossible subfunctor  $F \hookrightarrow \mathbf{B}(p(-), p(E))$  behaves like an object in  $\mathbf{E}$ .

**Definition 2.2.**

- Arbitrary subfunctor  $F \hookrightarrow \mathbf{B}(p(-), B) : \mathbf{E}^{op} \rightarrow \mathbf{Set}$  is called **almost-E structure** over object  $B \in Ob \mathbf{B}$ .

- Category  $\begin{smallmatrix} A\mathbf{E} \\ \downarrow \\ \mathbf{B} \end{smallmatrix}$  with objects  $\begin{pmatrix} F \\ B \end{pmatrix}$ ,  $B \in Ob \mathbf{B}$ ,  $F \hookrightarrow \mathbf{B}(p(-), B)$  and morphisms  $\begin{pmatrix} f \circ - \\ f \end{pmatrix} \equiv \begin{pmatrix} \mathbf{B}(p(-), f) \\ f \end{pmatrix}$ ,  $f : B \rightarrow B'$  is called a **category of almost-E structures over  $\mathbf{B}$** .

- **Almost-E costructure over  $B \in Ob \mathbf{B}$**  is a subfunctor  $F' \hookrightarrow \mathbf{B}(B, p(-)) : \mathbf{E} \rightarrow \mathbf{Set}$  [almost costructures are not dual to almost structures, they behave all together in a covariant way].

- Category  $\begin{smallmatrix} A'\mathbf{E} \\ \downarrow \\ \mathbf{B} \end{smallmatrix}$  with objects  $\begin{pmatrix} F' \\ B \end{pmatrix}$ ,  $B \in Ob \mathbf{B}$ ,  $F' \hookrightarrow \mathbf{B}(B, p(-))$  and morphisms  $f : \begin{pmatrix} F' \\ B \end{pmatrix} \rightarrow \begin{pmatrix} F'_1 \\ B_1 \end{pmatrix}$ , if  $f : B \rightarrow B_1 \in Ar \mathbf{B}$  and  $\forall E_1 \in Ob \mathbf{E}$ ,  $\forall g \in \mathbf{B}(B_1, p(E_1))$ .  $g \circ f \in \mathbf{B}(B, p(E_1))$ ,

is called a **category of almost-E costructures over  $\mathbf{B}$** .  $\square$

### Example

Take  $\mathbf{Poly}(E_1, E_2, \dots, E_n; -) \hookrightarrow \mathbf{Set}(p(E_1 \times \dots \times E_n), p(-)) : \mathbf{Vect} \rightarrow \mathbf{Set}$ , subfunctor of polylinear maps. Then  $\mathbf{Poly}(+, +, \dots, +; -) : \mathbf{Vect}^n \rightarrow \mathbf{Set}^{\mathbf{Vect}}$  determines a subcategory of almost- $\mathbf{Vect}$  costructures over  $(p \circ (\times^n))(\mathbf{Vect}^n) \hookrightarrow \mathbf{Set}$  ( $p : \mathbf{Vect} \rightarrow \mathbf{Set}$  is forgetful).



**Proposition 2.4.** For a structure of type  $\mathbf{E}$  on  $\mathbf{B}$

$$\begin{array}{c} \mathbf{E} \\ \downarrow p \\ \mathbf{B} \end{array}$$

- $\begin{array}{c} \mathbf{AE} \\ \downarrow \\ \mathbf{B} \end{array}$  is a fibration;
- $\begin{array}{c} \mathbf{E} \\ \downarrow p \\ \mathbf{B} \end{array} \hookrightarrow \begin{array}{c} \mathbf{AE} \\ \downarrow \\ \mathbf{B} \end{array}$  is a subcategory

*Proof.*

- If  $\begin{pmatrix} F \\ B \end{pmatrix} \in \text{Ob} \begin{array}{c} \mathbf{AE} \\ \downarrow \\ \mathbf{B} \end{array}$  and  $f : B' \rightarrow B$  take

$f^*F := \{g : p(X) \rightarrow B' \mid X \in \text{Ob } \mathbf{E}, f \circ g \in F(X) \subset \mathbf{B}(p(X), B)\}$ . Then  $f^*F \hookrightarrow \mathbf{B}(p(-), B')$

is a subfunctor, and  $\begin{pmatrix} f^*F \\ B' \end{pmatrix} \xrightarrow{\begin{pmatrix} f \circ - \\ f \end{pmatrix}} \begin{pmatrix} F \\ B \end{pmatrix}$  is cartesian over  $f$

$$\begin{array}{ccccc} F'' & & & & \\ & \searrow f \circ k \circ - & & & \\ B'' & \xrightarrow{k \circ -} & f^*F & \xrightarrow{f \circ -} & F \\ & \searrow f \circ k & & & \\ & & B' & \xrightarrow{f} & B \end{array}$$

- Assignment  $\left\{ \begin{array}{l} \begin{pmatrix} E \\ p(E) \end{pmatrix} \mapsto p(\mathbf{E}(-, E)) \hookrightarrow \mathbf{B}(p(-), p(E)) \quad \text{on objects} \\ \begin{pmatrix} v \\ p(v) \end{pmatrix} \mapsto p(\mathbf{E}(-, v)) = \mathbf{B}(p(-), p(v)) \quad \text{on arrows} \end{array} \right.$

gives the required embedding.  $\square$

### 3. Enrichment with generalized elements in hom-sets

**Definition 3.1.** 1-category  $\mathbf{C}$  is **enriched in tensor category**  $(\mathcal{V}, I, \otimes)$  [Kel, Bor2] if

- $\forall x, y \in \text{Ob } \mathbf{C} \quad \mathbf{C}(x, y) \in \text{Ob } \mathcal{V}$
- $\forall x, y, z \in \text{Ob } \mathbf{C} \quad \mu_{x,y,z} : \mathbf{C}(y, z) \otimes \mathbf{C}(x, y) \rightarrow \mathbf{C}(x, z) \in \text{Ar } \mathcal{V}$
- $\forall x, y, z, w \in \text{Ob } \mathbf{C}$

$$\begin{array}{ccc} (\mathbf{C}(z, w) \otimes \mathbf{C}(y, z)) \otimes \mathbf{C}(x, y) & \xrightarrow{\sim} & \mathbf{C}(z, w) \otimes (\mathbf{C}(y, z) \otimes \mathbf{C}(x, y)) \xrightarrow{1 \otimes \mu} \mathbf{C}(z, w) \otimes \mathbf{C}(x, z) \\ \mu \otimes 1 \downarrow & & \downarrow \mu \\ \mathbf{C}(w, y) \otimes \mathbf{C}(x, y) & \xrightarrow{\mu} & \mathbf{C}(x, w) \end{array}$$

- $\forall x \in \text{Ob } \mathbf{C} \exists u_x : I \rightarrow \mathbf{C}(x, x) \in \text{Ar } \mathcal{V}$  such that

$$\begin{array}{ccc} \mathbf{C}(x, y) & \xleftarrow{\sim} & I \otimes \mathbf{C}(x, y) \\ & \searrow \mu & \downarrow u_x \otimes 1 \\ & & \mathbf{C}(x, x) \otimes \mathbf{C}(x, y) \end{array} \quad \begin{array}{ccc} \mathbf{C}(z, x) & \xleftarrow{\sim} & \mathbf{C}(z, x) \otimes I \\ & \searrow \mu & \downarrow 1 \otimes u_x \\ & & \mathbf{C}(z, x) \otimes \mathbf{C}(x, x) \end{array}$$

$\square$

Generalized elements in hom-sets are just parametrized families of arrows in the same way as continuous or smooth families of maps. They form almost- $\mathbf{C}$  structure on hom-sets of category  $\mathbf{C}$  [Kom].

**Definition 3.2.** Assume,  $\mathbf{C}$  has binary products and  $|-| : \mathbf{C} \rightarrow \mathbf{Set}$  is a faithful functor. Then  $\mathbf{Set}$ -map  $f : |Z| \rightarrow \mathbf{C}(X, Y)$ ,  $Z \in \text{Ob } \mathbf{C}$  is called a **generalized** (or acceptable) **element** of  $\mathbf{C}(X, Y)$  with domain  $|Z|$  if arrow  $\tilde{f} \circ \gamma_Z$  can be lifted to  $\mathbf{C}$

$$\begin{array}{ccc} \mathbf{C}(X, Y) \times |X| & \xrightarrow{ev} & |Y| \\ f \times 1 \uparrow & \nearrow \tilde{f} & \\ |Z| \times |X| & & \\ \gamma_Z \uparrow & & \\ |Z \times X| & & \end{array}$$

where:  $\gamma_Z$  is the mediating arrow to product,  $ev(g, x) := |g|(x)$  □

Denote  $\mathcal{G}(|Z|, \mathbf{C}(X, Y)) \hookrightarrow \mathbf{Set}(|Z|, \mathbf{C}(X, Y))$  subset of generalized elements of  $\mathbf{C}(X, Y)$  with domain  $|Z|$ .

**Proposition 3.1.** *Assignment  $Z \mapsto \mathcal{G}(|Z|, \mathbf{C}(X, Y))$  is extendable to a functor  $\mathcal{G}(|-|, \mathbf{C}(X, Y)) : \mathbf{C}^{op} \rightarrow \mathbf{Set}$ .*

*Proof.* Assume,  $\alpha : Z' \rightarrow Z$  is an arrow,  $f \in \mathcal{G}(|Z|, \mathbf{C}(X, Y))$  is a generalized element. We need to show that  $f \circ \alpha \in \mathcal{G}(|Z'|, \mathbf{C}(X, Y))$  is a generalized element as well, i.e., that  $\exists h : Z' \times X \rightarrow Y$  such that  $|h| = ev \circ (f \times 1) \circ (\alpha \times 1) \circ \gamma_{Z'} = \tilde{f} \circ (\alpha \times 1) \circ \gamma_{Z'}$ .

$$\begin{array}{ccc} \text{Since } \gamma_Z : |Z \times X| \rightarrow |Z| \times |X| \text{ is natural in } Z & |Z' \times X| & \xrightarrow{\gamma_{Z'}} |Z'| \times |X| \\ & \downarrow |\alpha \times 1| & \downarrow |\alpha| \times |1| \\ & |Z \times X| & \xrightarrow{\gamma_Z} |Z| \times |X| \xrightarrow{\tilde{f}} |Y| \end{array}$$

the requirement will be  $\exists h : Z' \times X \rightarrow Y$  such that  $|h| = \tilde{f} \circ \gamma_Z \circ |\alpha \times 1|$ . By assumption on  $f$ ,  $\exists g : Z \times X \rightarrow Y$  such that  $|g| = \tilde{f} \circ \gamma_Z$ . So, take  $h := g \circ (\alpha \times 1)$ . □

**Proposition 3.2.** *If  $|-| : \mathbf{C} \rightarrow \mathbf{Set}$  is a faithful functor which preserves binary products, then category  $\mathbf{C}$  is enriched with generalized elements in presheaves category  $\mathbf{Set}^{\mathbf{C}^{op}}$ .*

*Proof.*

- $\forall X, Y \in \text{Ob } \mathbf{C} \quad (\mathcal{G}(|-|, \mathbf{C}(X, Y)) : \mathbf{C}^{op} \rightarrow \mathbf{Set}) \in \text{Ob}(\mathbf{Set}^{\mathbf{C}^{op}})$
- $\forall X, Y, Z \in \text{Ob } \mathbf{C}$  take  $\mu_{X,Y,Z} : \mathcal{G}(|-|, \mathbf{C}(Y, Z)) \otimes \mathcal{G}(|-|, \mathbf{C}(X, Y)) \rightarrow \mathcal{G}(|-|, \mathbf{C}(X, Z))$  such that  $\forall W \in \text{Ob } \mathbf{C} \quad \mu_{X,Y,Z;W}(f, g) := \mu_{X,Y,Z}^{\mathbf{C}} \circ \langle f, g \rangle$  where:  $\mu_{X,Y,Z}^{\mathbf{C}} : \mathbf{C}(Y, Z) \times \mathbf{C}(X, Y) \rightarrow \mathbf{C}(X, Z)$  is the composite in  $\mathbf{C}$ ,  $\langle f, g \rangle : |W| \rightarrow \mathbf{C}(Y, Z) \times \mathbf{C}(X, Y)$  is the mediating arrow to product.  $\mu_{X,Y,Z;W}$  is natural in  $W$  since  $(\mu_{X,Y,Z}^{\mathbf{C}} \circ \langle f, g \rangle) \circ |h| = \mu_{X,Y,Z}^{\mathbf{C}} \circ \langle f \circ |h|, g \circ |h| \rangle$  for  $h : W' \rightarrow W$ .

Why  $ev \circ ((\mu^{\mathbf{C}} \circ \langle f, g \rangle) \times 1) \circ \gamma$  can be lifted to  $\mathbf{C}$ ? By condition,

$$\begin{array}{ccc}
\mathbf{C}(Y, Z) \times |Y| & \xrightarrow{ev} & |Z| \\
\uparrow f \times 1 & \nearrow \tilde{f} & \\
|W| \times |Y| & \xrightarrow{\text{lifted}} & \\
\uparrow \gamma & & \\
|W \times Y| & & 
\end{array}
\qquad
\begin{array}{ccc}
\mathbf{C}(X, Y) \times |X| & \xrightarrow{ev} & |Y| \\
\uparrow g \times 1 & \nearrow \tilde{g} & \\
|W| \times |X| & \xrightarrow{\text{lifted}} & \\
\uparrow \gamma & & \\
|W \times X| & & 
\end{array}$$

Sufficient to take  $\gamma = 1$ .

$$\begin{array}{ccccc}
& \mathbf{C}(X, Z) \times |X| & \xrightarrow{\quad \dots \quad} & |Z| & \\
& \uparrow \mu^{\mathbf{C}} \times 1_{|X|} & & \uparrow ev & \\
\mathbf{C}(Y, Z) \times \mathbf{C}(X, Y) \times |X| & \xrightarrow{1_{\mathbf{C}(Y, Z)} \times ev} & \mathbf{C}(Y, Z) \times |Y| & & \\
& \nwarrow f \times g \times 1_{|X|} & \uparrow f \times \tilde{g} = (f \times 1_{|Y|}) \circ (1_{|W|} \times \tilde{g}) & \nearrow \tilde{f} \circ (1_{|W|} \times \tilde{g}) & \\
& \nwarrow \langle f, g \rangle \times 1_{|X|} & |W| \times |W| \times |X| & & \\
& & \uparrow \langle 1_{|W|}, 1_{|W|} \rangle \times 1_{|X|} & & \\
& & |W| \times |X| & & \\
& & \uparrow \gamma = 1 & & \\
& & |W \times X| & & 
\end{array}$$

The required dotted way  $ev \circ ((\mu^{\mathbf{C}} \circ \langle f, g \rangle) \times 1_{|X|}) \circ \gamma$  can be lifted to  $\mathbf{C}$  since the most right way  $\tilde{f} \circ (1_{|W|} \times \tilde{g}) \circ (\langle 1_{|W|}, 1_{|W|} \rangle \times 1_{|X|}) \circ 1_{|W \times X|}$  can be lifted (for take liftings for  $\tilde{f}, \tilde{g}$  and identities for identities).

- (associativity)  $\forall f, g, h$  such that  $f : |W| \rightarrow \mathbf{C}(Z, Z'), g : |W| \rightarrow \mathbf{C}(Y, Z), h : |W| \rightarrow \mathbf{C}(X, Y)$   $\mu_{X, Z, Z'}^{\mathbf{C}} \circ \langle f, \mu_{X, Y, Z}^{\mathbf{C}} \circ \langle g, h \rangle \rangle = \mu_{X, Y, Z'}^{\mathbf{C}} \circ \langle \mu_{Y, Z, Z'}^{\mathbf{C}} \circ \langle f, g \rangle, h \rangle$  because the equality holds at each point  $w \in |W|$ .
- (identities)  $\forall X \in Ob \mathbf{C}$  take  $u_{X, W} : \mathbf{1} \rightarrow \mathbf{Set}(|W|, \mathbf{C}(X, X)) : * \mapsto (w \mapsto 1_X)$ . It is natural in  $W$  and  $\forall f, g$  such that  $f : |W| \rightarrow \mathbf{C}(X, Y), g : |W| \rightarrow \mathbf{C}(Z, X)$  the equalities hold  $\mu^{\mathbf{C}} \circ \langle f, u_{X, W} \rangle (w) = \mu^{\mathbf{C}}(f(w), 1_X) = f(w), \mu^{\mathbf{C}} \circ \langle u_{X, W}, g \rangle (w) = \mu^{\mathbf{C}}(1_X, g(w)) = g(w) \ w \in |W|$ .  $\square$

**Corollary** (refinement of Proposition 3.2). *Under the above assumptions ( $\mathbf{C}$  has binary products,  $|-| : \mathbf{C} \rightarrow \mathbf{Set}$  is a faithful functor preserving binary products) hom-sets of  $\mathbf{C}$  enriched with almost- $\mathbf{C}$  structure of generalized elements.*

*Proof* is immediate (because presheaves of generalized elements are of specific form  $\mathcal{G}(|-, \mathbf{C}(X, Y)|) \hookrightarrow \mathbf{Set}(|-, \mathbf{C}(X, Y)|)$  and  $\mu$  is actually postcomposite  $\mu^{\mathbf{C}} \circ -$ ).  $\square$

We mean further that  $\mathbf{C}$  is  $\mathbf{AC}$ -category if this **specific enrichment** with generalized elements is given. Moreover, we call  $\mathbf{D}$  is  $\mathbf{AC}$ -category if it is enriched with presheaves of generalized elements with domains in  $\mathbf{C}$ .

**Remark.** All usual concrete categories, like **Top**, **Grp**, **Rng**, etc. carry corresponding almost structures (which in some cases can be strict).

### Example

**Proposition 3.3.** *If  $X$  is a locally compact topological space (so that, family  $\mathcal{T}$  of topologies on  $\mathbf{Top}(X, Y)$  for which evaluation map  $ev : \mathbf{Top}(X, Y) \times |X| \rightarrow |Y|$  is continuous is not empty and contains minimal element, compact-open topology on  $\mathbf{Top}(X, Y)$ ) then  $\tau \in \mathcal{T}$  is compact-open iff  $\forall Z \in Ob \mathbf{Top}$  each generalized element  $f : |Z| \rightarrow \mathbf{Top}(X, Y)$  is continuous.*

*Proof.* "  $\Rightarrow$  " Regard the diagram

$$\begin{array}{ccc} \mathbf{Top}(X, Y) \times |X| & \xrightarrow{ev} & |Y| \\ f \times 1 \uparrow & \nearrow \tilde{f} & \\ |Z| \times |X| & & \end{array}$$

We want to show that  $f : |Z| \rightarrow \mathbf{Top}(X, Y)$  is continuous (with compact-open topology in  $\mathbf{Top}(X, Y)$ ) if  $\tilde{f} : |Z| \times |X| \rightarrow |Y|$  is continuous, i.e., that  $\forall z \in |Z| \ \forall$  (subbase) compact-open set  $U^K \ni f(z) \ \exists$  nbhd  $V \ni z$  such that  $f(V) \subset U^K$ . It is equivalent that  $\forall z \in |Z| \ \forall U^K \ni f(z) \ \exists V \ni z$  such that  $\tilde{f}(V \times K) \subset U$ . Since  $\tilde{f}$  is continuous  $\forall (z, x) \in \{z\} \times K$  and  $\forall$  open  $U \ni \tilde{f}(z, x) \ \exists$  open nbhd  $V_z \times W_x \ni (z, x)$  such that  $\tilde{f}(V_z \times W_x) \subset U$ .  $\bigcup_{x \in K} W_x \supset K$  (open cover). So, by compactness

of  $K$ ,  $\exists W_{x_1}, \dots, W_{x_n}$ , such that  $\bigcup_{i=1}^n W_{x_i} \supset K$ . Take  $V := \bigcap_{i=1}^n (V_i)_z$ , where  $(V_i)_z$  corresponds to  $W_{x_i}$  (i.e.,  $(V_i)_z$  is open,  $(V_i)_z \ni z$ ,  $\tilde{f}((V_i)_z \times W_{x_i}) \subset U$ ). Then  $\tilde{f}(V \times K) \subset U$ .

"  $\Leftarrow$  " Take  $Z = \mathbf{Top}(X, Y)$  with compact-open topology. Take  $\mathbf{Top}(X, Y)$  itself (on the top of the diagram) with non minimal  $\tau \in \mathcal{T}$ ,  $f := 1 \in Ar \mathbf{Set}$ . Then  $1 : \mathbf{Top}(X, Y) \rightarrow \mathbf{Top}(X, Y)$  is an admissible generalized element, since  $ev$  is continuous, but  $1$  is not continuous.  $\square$

**Remark.** Therefore, for locally compact space  $X$  almost-**Top** structure  $\mathcal{G}(|Z|, \mathbf{Top}(X, Y))$  coincides with compact-open topology, i.e., is actually **Top** structure.

If we agree that functor  $\mathcal{G}(|-, \mathbf{C}(X, Y)) : \mathbf{C}^{op} \rightarrow \mathbf{Set}$  reflects essential properties of **C**-hom-sets we immediately get a unique (up to isomorphism) extension of each functor  $F : \mathbf{C} \rightarrow \mathbf{D}$ , i.e., deal with **C**-hom-sets as with **C**-objects. In this way, for example, tangent or jet functor can be introduced directly on  $\mathbf{Aut}(X)$ ,  $X \in Ob \mathbf{Diff}$  to give rise a calculus on  $\mathbf{Aut}(X)$ . Possibility of such an extension follows from the fact that each presheaf is a certain (canonical) colimit of representables [Mac, M-M].

### Proposition 3.4.

- Yoneda embedding  $y : \mathbf{C} \rightarrow \mathbf{Set}^{\mathbf{C}^{op}}$  is a universal cocompletion of **C**, i.e.,  $\forall F : \mathbf{C} \rightarrow \mathbf{E}$ , where

$$\begin{array}{ccc} \mathbf{Set}^{\mathbf{C}^{op}} & \xrightarrow{\bar{F}} & \mathbf{E} \\ y \uparrow & \nearrow F & \\ \mathbf{C} & & \end{array}$$

$\mathbf{E}$  is cocomplete,  $\exists!$  (up to iso) cocontinuous  $\bar{F} : \mathbf{Set}^{\mathbf{C}^{op}} \rightarrow \mathbf{E}$  such that

- $\bar{F}(P) = \text{Colim}(\int_{\mathbf{C}} P \xrightarrow{\pi} \mathbf{C} \xrightarrow{F} \mathbf{E})$ , where  $P \in Ob \mathbf{Set}^{\mathbf{C}^{op}}$ ,  $\int_{\mathbf{C}} P$  is a category of elements of  $P$ ,  $\pi$  is the natural projection.

- $\mathbf{Cat} \xrightleftharpoons[\text{forgetful}]{\mathbf{Set}^{(-)^{op}}} \mathbf{Cocomp}$ , adjunction between  $\mathbf{Cat}$  and full subcategory of cocomplete categories  $\mathbf{Cocomp}$  with Yoneda embedding  $y_{\mathbf{C}} : \mathbf{C} \rightarrow \mathbf{Set}^{\mathbf{C}^{op}}$  as a unit.
- Each functor  $F : \mathbf{C} \rightarrow \mathbf{D}$  admits a unique (up to iso) cocontinuous extension  $F : \mathbf{Set}^{\mathbf{C}^{op}} \rightarrow \mathbf{Set}^{\mathbf{D}^{op}}$  such that
 
$$\begin{array}{ccc} \mathbf{Set}^{\mathbf{C}^{op}} & \xrightarrow{F} & \mathbf{Set}^{\mathbf{D}^{op}} \\ y_{\mathbf{C}} \uparrow & & \uparrow y_{\mathbf{D}} \\ \mathbf{C} & \xrightarrow{F} & \mathbf{D} \end{array}$$

*Proof.* See [M-M].  $\square$

For example, if  $T : \mathbf{Diff} \rightarrow \mathbf{Diff}$  is a tangent functor,  $\mathbf{Diff}(X, Y)$  is a presheaf on  $\mathbf{Diff}$  (hom-set enriched as above) then  $T(\mathbf{Diff}(X, Y)) = \int_{\mathbf{Diff}} \mathbf{Diff}(X, Y) \xrightarrow{\pi} \mathbf{Diff} \xrightarrow{T} \mathbf{Diff} \xrightarrow{y} \mathbf{Set}^{\mathbf{Diff}^{op}}$ .

### 3.1. Tangent functor for smooth algebras.

It is an example of dual (and invariant) construction for the main functor of Differential Geometry (which gives suggestion how it can be extended over spectra of commutative algebras).

Let  $T : \mathbf{Diff} \rightarrow \mathbf{Diff}$  be **tangent** functor on the category of real  $\infty$ -smooth manifolds. In local

$$\text{coordinates it looks like } \begin{cases} X \rightarrow TX : (x^i) \rightarrow (x^i, \xi^j) & X \in Ob \mathbf{Diff} \\ f \rightarrow Tf : (f^i(x)) \rightarrow (f^i(x), \frac{\partial f^j}{\partial x^k} \xi^k) & f \in Ar \mathbf{Diff} \end{cases}$$

$\mathbf{Diff} \hookrightarrow \mathbb{R}\text{-}\mathbf{Alg}^{op}$  is a subcategory of the opposite of real commutative algebras. Working in  $\mathbf{Diff}$  it is hard (if possible at all) to give coordinate-free characterization of  $T$ . The question is how it looks like in  $\mathbb{R}\text{-}\mathbf{Alg}$ ?

**Definition 3.1.1.** Let  $\mathcal{A} \in Ob \mathbb{R}\text{-}\mathbf{Alg}$ .

- $\rho : \mathcal{A} \rightarrow \mathbf{Top}(\mathbf{Spec}_{\mathbb{R}}(\mathcal{A}), \mathbb{R})$  is called **functional representation** homomorphism of  $\mathcal{A}$ , where  $\mathbf{Spec}_{\mathbb{R}}(\mathcal{A}) = \mathbb{R}\text{-}\mathbf{Alg}(\mathcal{A}, \mathbb{R})$  with initial topology w.r.t. all functions  $\rho(a)$ ,  $a \in \mathcal{A}$ ,  $\rho(a)(f) := ev(f, a) := |f|(a)$ .
- $\mathcal{A}$  is called **smooth** if  $\forall a_1, a_2, \dots, a_n \in \mathcal{A}$  and  $\forall f : \mathbb{R}^n \rightarrow \mathbb{R} \in C^\infty(\mathbb{R}^n)$  the composite  $f \circ \langle \rho(a_1), \rho(a_2), \dots, \rho(a_n) \rangle \in \text{Im}(\rho)$ .  $\square$

Denote by  $\mathbb{R}\text{-}\mathbf{Sm}\text{-}\mathbf{Alg} \hookrightarrow \mathbb{R}\text{-}\mathbf{Alg}$  full subcategory of smooth algebras.

**Lemma 3.1.1.**  $\mathbb{R}\text{-}\mathbf{Sm}\text{-}\mathbf{Alg} \hookrightarrow \mathbb{R}\text{-}\mathbf{Alg}$  is a reflective subcategory, i.e., the inclusion has a left adjoint  $\mathbf{Sm} : \mathbb{R}\text{-}\mathbf{Alg} \rightarrow \mathbb{R}\text{-}\mathbf{Sm}\text{-}\mathbf{Alg}$ , **smooth completion** of  $\mathbb{R}$ -algebras.

*Proof.* Just take for each  $\mathbb{R}$ -algebra  $\mathcal{A}$   $\mathbf{Sm}(\mathcal{A})$  of all terms  $\{f(a_1, \dots, a_n) \mid f : \mathbb{R}^n \rightarrow \mathbb{R}, a_1, \dots, a_n \in \mathcal{A}\}$  (all smooth operations are admitted). Each morphism  $f$  from an  $\mathbb{R}$ -algebra  $\mathcal{A}$  to a smooth algebra  $\mathcal{B}$  is uniquely extendable to  $\tilde{f} : \mathbf{Sm}(\mathcal{A}) \rightarrow \mathcal{B}$ .  $\square$

Let  $\mathbf{Sym}\text{-}\mathbf{Alg}$  be a category of symmetric partial differential algebras.  $Ob(\mathbf{Sym}\text{-}\mathbf{Alg})$  are graded commutative algebras over commutative  $\mathbb{R}$ -algebras with a differential  $d : \mathcal{A}^0 \rightarrow \mathcal{A}^1$  of degree 1 determined only on elements of degree 0 ( $d$  is  $\mathbb{R}$ -linear and satisfies Leibniz rule).  $Ar(\mathbf{Sym}\text{-}\mathbf{Alg})$  are graded degree 0 algebra homomorphisms which respect  $d$ .

**Lemma 3.1.2.** There is an adjunction  $\mathbb{R}\text{-}\mathbf{Alg} \xrightleftharpoons[p_0]{\mathbf{Sym}} \mathbf{Sym}\text{-}\mathbf{Alg}$ ,

$$\text{where: } p_0 \text{ is the projection onto 0-component } \begin{cases} p_0(\mathcal{A}) := \mathcal{A}^0 \\ p_0(\mathcal{A} \xrightarrow{f} \mathcal{B}) := (\mathcal{A}^0 \xrightarrow{f^0} \mathcal{B}^0) \end{cases},$$

**Sym** is taking symmetric algebra over module of differentials of the given algebra

$$\begin{cases} \mathbf{Sym}(\mathcal{C}) := \mathbf{Sym}(\Lambda^1(\mathcal{C})) \\ \mathbf{Sym}(\mathcal{C} \xrightarrow{h} \mathcal{D}) := (\mathbf{Sym}(\mathcal{C}) \xrightarrow{\tilde{h}} \mathbf{Sym}(\mathcal{D})) \\ \tilde{h}(\sum c_{i^1 \dots i^k} (dc_1)^{i_1} \dots (dc_k)^{i_k}) := \sum h(c_{i^1 \dots i^k}) (dh(c_1))^{i_1} \dots (dh(c_k))^{i_k} \end{cases}$$

□

**Lemma 3.1.3.** 
$$\begin{array}{ccc} \mathbb{R}\text{-}\mathbf{Alg} & \xrightarrow{\mathbf{Sm}} & \mathbb{R}\text{-}\mathbf{Sm}\text{-}\mathbf{Alg} \\ & \searrow \mathbf{Spec}_{\mathbb{R}} & \downarrow \mathbf{Spec}_{\mathbb{R}} \\ & & \mathbf{Top} \end{array} \quad (\text{smooth completion does not change spectrum}).$$

*Proof.*  $\forall \alpha : \mathcal{A} \rightarrow \mathbb{R} \exists!$  extension  $\tilde{\alpha} : \mathbf{Sm}(\mathcal{A}) \rightarrow \mathbb{R} : f(a_1, \dots, a_n) \mapsto f(\alpha(a_1), \dots, \alpha(a_n))$ . And conversely, each such  $\tilde{\alpha}$  is uniquely restricted to  $\alpha$ . Initial topology on  $\mathbb{R}\text{-}\mathbf{Alg}((\mathbf{Sm})(\mathcal{A}), \mathbb{R})$  does not change because new functions are functionally dependent on old ones. □

**Remark.** With Zariski topology in spectra smooth completion yields the same set with a weaker topology. For  $C^\infty(X)$ ,  $X \in \mathbf{Ob} \mathbf{Diff}$  Zariski and initial topologies coincide.

**Proposition 3.1.1.** • *Tangent functor  $T : \mathbb{R}\text{-}\mathbf{Sm}\text{-}\mathbf{Alg} \rightarrow \mathbb{R}\text{-}\mathbf{Sm}\text{-}\mathbf{Alg}$  is equal to the composite  $\mathbb{R}\text{-}\mathbf{Sm}\text{-}\mathbf{Alg} \hookrightarrow \mathbb{R}\text{-}\mathbf{Alg} \xrightarrow{\mathbf{Sym}} \mathbf{Sym}\text{-}\mathbf{Alg} \xrightarrow{U} \mathbb{R}\text{-}\mathbf{Alg} \xrightarrow{\mathbf{Sm}} \mathbb{R}\text{-}\mathbf{Sm}\text{-}\mathbf{Alg}$ , where  $U$  forgets differential  $d$  and grading.*

$$\bullet \text{ To canonical projection } p_X : TX \downarrow X \text{ there corresponds canonical embedding } i_{C^\infty(X)} : C^\infty(X) \uparrow T(C^\infty(X)).$$

*Proof.*

- If  $X \in \mathbf{Ob} \mathbf{Diff}$   $TX \sim \mathbf{Spec}_{\mathbb{R}}(U \circ \mathbf{Sym}(C^\infty(X))) \sim \mathbf{Spec}_{\mathbb{R}}(\mathbf{Sm} \circ U \circ \mathbf{Sym}(C^\infty(X)))$ .
- immediate. □

**Remark.** It is reasonable to define  $T$  on  $\mathbb{R}\text{-}\mathbf{Alg}$  as  $T := U \circ \mathbf{Sym}$  and transfer it to spectra via duality  $\mathbb{R}\text{-}\mathbf{Alg}^{op} \xrightleftharpoons[\perp]{F} \mathbf{Spec}_{\mathbb{R}}$  (as  $F \circ T^{op} \circ G$ ).

#### 4. General manifolds

**Definition 4.1.** Functor  $F : \mathbf{E} \downarrow \mathbf{B}$  is called a **fibration with respect to** class of arrows  $\mathcal{C} \subset \mathbf{Ar} \mathbf{B}$

if  $\forall f : B' \rightarrow F(E) \in \mathcal{C} \exists \tilde{f} : E' \rightarrow E \in \mathbf{Ar} \mathbf{E}$  such that  $\tilde{f}$  is over  $f$  and  $\tilde{f}$  is cartesian. □

**Definition 4.2.** [M-M]

- Grothendieck **pretopology**  $\tau_0$  on a category  $\mathbf{B}$  with pullbacks is a family of **coverings**  $\tau_{0B}$  for each object  $B \in \mathbf{Ob} \mathbf{B}$  (elements of a covering are just arrows with codomain  $B$ ) such that
  - if  $f : B' \xrightarrow{\sim} B$  is an iso then  $\{f\} \in \tau_{0B}$  is an one-element covering
  - if  $g : B'' \rightarrow B$  is an arrow and  $\mathbf{c} \in \tau_{0B}$  then pullback family  $\mathbf{plbk}_g(\mathbf{c}) \in \tau_{0B''}$
  - (coverings are composable) if  $\mathbf{c} \in \tau_{0B}$  and  $\forall B' \in d(\mathbf{c})$  there is given a covering  $\mathbf{c}_{B'} \in \tau_{0B'}$  then  $\mathbf{c} \circ \bigcup_{B' \in d(\mathbf{c})} \mathbf{c}_{B'} \in \tau_{0B}$

- Grothendieck **topology**  $\tau$  on a category  $\mathbf{B}$  (not necessarily with pullbacks) is a family of hom-subfunctors  $\tau_B$  for each object  $B \in \text{Ob } \mathbf{B}$  such that
  - $\mathbf{B}(-, B) \in \tau_B$
  - if  $f : B' \rightarrow B$  and  $\mathbf{t} \in \tau_B$  then the **inverse image**  $(f^*(\mathbf{t}) : X \mapsto (f^*\mathbf{t})(X, B') \subset \mathbf{B}(X, B')) \in \tau_{B'}$  ( $h \in f^*(\mathbf{t})(X, B')$  iff  $f \circ h \in \mathbf{t}(X, B)$ )
  - if  $\mathbf{s} \hookrightarrow \mathbf{B}(-, B)$  is any hom-subfunctor such that  $\forall f : B' \rightarrow B$   $f^*(\mathbf{s}) \in \tau_{B'}$  then  $\mathbf{s} \in \tau_B$   $\square$

Every topology is a pretopology if  $\mathbf{B}$  has pullbacks, and every pretopology generates a topology [M-M]. Category  $\mathbf{B}$  with (pre)topology is called **site**  $(\mathbf{B}, \tau_0)$ .

**Definition 4.3.** [Kom] Functor 
$$\begin{array}{c} \mathbf{E} \\ \downarrow F \\ (\mathbf{B}, \tau_0) \end{array}$$
 is called **local** if it is a fibration with respect to all

elements of all coverings  $\bigcup_{B \in \text{Ob } \mathbf{B}} \tau_0 B$ .  $\square$

**Definition 4.4.** For a given functor to a site 
$$\begin{array}{c} \mathbf{E} \\ \downarrow F \\ (\mathbf{B}, \tau_0) \end{array}$$
 smallest local functor 
$$\begin{array}{ccc} \mathbf{E-Man} & \xleftarrow{\quad} & \mathbf{E} \\ \downarrow p_F & \swarrow F & \\ (\mathbf{B}, \tau_0) & & \end{array}$$

is called **E-manifold structure over B**. It means

- $\forall X \in \text{Ob } \mathbf{E-Man} \exists$  covering  $\mathbf{c}_{p_F(X)} = \{i : B_i \rightarrow p_F(X)\}_{i \in I} \in \tau_{0p_F(X)}$  such that there are inverse images  $i^*(X) \in \text{Ob } \mathbf{E}$  (i.e.,  $\mathbf{E}$  contains isomorphic representatives of the inverse images)
- $\forall f : X' \rightarrow X \in \text{Ar } \mathbf{E-Man} \exists$  coverings  $\mathbf{c}'_{p_F(X')} = \{i' : B'_{i'} \rightarrow p_F(X')\} \in \tau_{0p_F(X')}$  and  $\mathbf{c}_{p_F(X)} = \{i : B_i \rightarrow p_F(X)\} \in \tau_{0p_F(X)}$  such that  $\forall i \in \mathbf{c}_{p_F(X)} \exists i' \in \mathbf{c}'_{p_F(X')}$  such that

$$\begin{array}{ccc} B'_{i'} & \xrightarrow{\exists \varphi} & B_i \\ i' \downarrow & & \downarrow i \\ p_F(X') & \xrightarrow[p_F(f)]{} & p_F(X) \end{array} \quad \text{and over it} \quad \begin{array}{ccc} i'^*(X') & \xrightarrow{\exists! \Phi} & i^*(X) \\ \tilde{i}' \downarrow & & \downarrow \tilde{i} \\ X' & \xrightarrow{f} & X \end{array}$$

$\tilde{i}', \tilde{i}$  are cartesian,  $p_F(\Phi) = \varphi$ ,  $\Phi \in \text{Ar } \mathbf{E}$  (arrows are locally in  $\mathbf{E}$ )

- **E-Man** is maximal with respect to two above properties  $\square$

### Examples

1. **Set** as a manifold structure 
$$\begin{array}{ccc} \mathbf{Set} & \xleftarrow{\quad} & \mathbf{Set}_{inj} \\ 1 \downarrow & \swarrow & \\ (\mathbf{Set}, \tau_0) & & \end{array}$$
 where  $\mathbf{Set}_{inj}$  is a category of sets with

injective maps only,  $\tau_0$  is a pretopology consisting of all families of injective maps with common codomains.

2. **Differentiable** manifolds 
$$\begin{array}{ccc} C^r\text{-Man}_k & \xleftarrow{\quad} & \text{Triv } C^r\text{-Man}_k \\ \downarrow & \swarrow & \\ (\mathbf{Top}, \tau_0) & & \end{array}$$
 where  $k = \mathbb{R}$  or  $\mathbb{C}$ ,

$\tau_0$  consists of all open coverings,  $r = 0, 1, \dots, \infty$  (or  $\omega$  for complex manifolds),

$$\begin{cases} Ob(\mathbf{Triv}C^r\text{-}\mathbf{Man}_k) = \{k^0, k^1, \dots, k^n, \dots\} \\ Ar(\mathbf{Triv}C^r\text{-}\mathbf{Man}_k) = C^r\text{-maps} \end{cases}$$

3. Locally trivial **fibre bundles**

$$\begin{array}{ccc} \mathbf{Bn}(\mathcal{E}, p) & \xleftarrow{\quad} & \mathbf{Bn}_0(\mathcal{E}, p) \\ \downarrow & \swarrow & \\ (\mathbf{Man}^{\rightarrow}, \tau_0) & & \end{array} \quad (\text{see 5})$$

4. **Foliations** over **Man**

$$\begin{array}{ccc} \mathbf{Fol}(\mathcal{E}, p) & \xleftarrow{\quad} & \mathbf{Fol}_0(\mathcal{E}, p) \\ \downarrow & \swarrow & \\ (\mathbf{Man}, \tau_0) & & \end{array} \quad \text{associated to } A\mathbf{Man}\text{-functor sequence}$$

$\mathcal{E} \xrightarrow{p} \mathbf{Man} \xrightarrow{\pi} \mathbf{Top}$  (see 5), where: **Man** is a category of manifolds (of type  $\mathcal{E}'$ ) over **Top**,  $\tau_0$  are all 'open coverings' of objects in **Man**,  $\mathbf{Fol}_0(\mathcal{E}, p) \equiv \mathbf{Bn}_0(\mathcal{E}, p)$  is a category of trivial

foliations ('direct products') with leaves in  $\mathcal{E}$ , projection functor  $\mathbf{Fol}_0(\mathcal{E}, p) \downarrow (\mathbf{Man}, \tau_0)$  is the first (top)

component of projection  $\mathbf{Bn}_0(\mathcal{E}, p) \downarrow (\mathbf{Man}^{\rightarrow}, \tau_0)$  ( $\tau_0$  are different in these two cases and corresponding categories of manifolds are glued differently).

5. **E**-manifolds over **Top**. Let  $\begin{array}{c} \mathbf{E} \\ p \downarrow \\ (\mathbf{Top}, \tau_0) \end{array}$  be a local structure on **Top** with  $\tau_0$ , all open coverings.

- **Local E-map** on a topological space  $X$  is a pair  $\left( \begin{array}{c} E \\ U \end{array} \right) \in Ob \begin{array}{c} \mathbf{E} \\ \downarrow \\ (\mathbf{Top}, \tau_0) \end{array}, \quad U \text{ is open.}$
- Family  $\left\{ \left( \begin{array}{c} E_i \\ U_i \end{array} \right) \right\}_{i \in I}$  is **compatible** iff  $\forall (i, j) \in I^2 \quad E_i|_{U_i \cap U_j} \xrightarrow[\varphi]{\sim} U_j|_{U_i \cap U_j}$ ,  $\varphi$  is a vertical iso.
- **E-atlas**  $\mathcal{A}$  on  $X$  is a compatible family  $\left\{ \left( \begin{array}{c} E_i \\ U_i \end{array} \right) \right\}_{i \in I}$  such that  $\bigcup_{i \in I} U_i = X$ .
- Two **E**-atlases  $\mathcal{A}$  and  $\mathcal{A}'$  are **equivalent** iff  $\mathcal{A} \cup \mathcal{A}'$  is still an **E**-atlas on  $X$  (so, there exist maximal atlases, call them  $\mathcal{A}_{max}$ ,  $\mathcal{B}_{max}$ , etc.).
- The above 'equivalence' on atlases is not transitive in general. So, there can be different maximal atlases containing a given one. But, it is transitive if  $\forall \left( \begin{array}{c} E \\ U \end{array} \right), \left( \begin{array}{c} E' \\ U \end{array} \right) \in Ob \begin{array}{c} \mathbf{E} \\ p \downarrow \\ \mathbf{Top} \end{array}$  and  $\forall$  open covering  $\bigcup_{i \in I} U_i \supset U \quad E|_{U_i} \xrightarrow[\text{vert}]{\sim} E'|_{U_i}$  (for all  $i \in I$ ) implies  $E \xrightarrow[\text{vert}]{\sim} E'$ .
- Topological space  $X$  together with an **E**-atlas  $\mathcal{A}$  on it is called **E-manifold**, i.e.,  $(X, \mathcal{A}) \in$



**Ob E-Man.**

- An **arrow** in **E-Man** is  $f : (X, \mathcal{A}) \rightarrow (Y, \mathcal{B})$  such that  $f : X \rightarrow Y$  is continuous and  $\forall \begin{pmatrix} E \\ U \end{pmatrix} \in \mathcal{A}, \begin{pmatrix} E' \\ V \end{pmatrix} \in \mathcal{B}$  if  $U \cap f^{-1}(V) \neq \emptyset$  then  $f|_{U \cap f^{-1}(V)} : U \cap f^{-1}(V) \rightarrow V$  admits (unique) lifting  $\bar{f}|_{U \cap f^{-1}(V)} : E|_{U \cap f^{-1}(V)} \rightarrow E' \in \text{Ar } \mathbf{E}$ .

## 5. Fibre bundles

Locally trivial fibre bundles give an important example of general manifolds over  $\mathbf{Man}^\rightarrow$  [Kom].

**Definition 5.1.** Category of **trivial fibre bundles**  $\mathbf{Bn}_0(\mathcal{E}, p)$  over  $\mathbf{Man}^\rightarrow$  with **typical fibres** in a category  $\mathcal{E}$  consists of the following data

- $\mathcal{E} \xrightarrow{p} \mathbf{Man} \xrightarrow{\pi} \mathbf{Top}$ , where  $\mathcal{E}$  and  $\mathbf{Man}$  are **AMan**-categories,  $p$  is **AMan**-functor,  $\pi$  preserves binary products [i.e.,  $\mathbf{Man}$  is enriched in  $\mathbf{Set}^{\mathbf{Man}^{op}}$  with presheaves of generalized elements  $\mathcal{G}(|-|, \mathbf{Man}(A, A'))$  for each hom-set  $\mathbf{Man}(A, A')$ ,  $\mathcal{E}$  is enriched in  $\mathbf{Set}^{\mathbf{Man}^{op}}$  with subfunctors  $\mathcal{H}(|-|, \mathcal{E}(E, E')) \hookrightarrow \mathbf{Set}(|-|, \mathcal{E}(E, E'))$  for each hom-set  $\mathcal{E}(E, E')$ ,  $p_{E, E'; X} : \mathcal{H}(|X|, \mathcal{E}(E, E')) \rightarrow \mathcal{G}(|X|, \mathbf{Man}(p(E), p(E')))) : f \mapsto p_{E, E'} \circ f$  is natural in  $X \in \text{Ob } \mathbf{Man}$ ,  $p_{E, E'} : \mathcal{E}(E, E') \rightarrow \mathbf{Man}(p(E), p(E'))$  is the restriction of functor  $p$  on the hom-set]
- $\text{Ob } \mathbf{Bn}_0(\mathcal{E}, p) := \{(X, E) \mid X \in \text{Ob } \mathbf{Man}, E \in \text{Ob } \mathcal{E}\};$   
 $\text{Ar } \mathbf{Bn}_0(\mathcal{E}, p) := \{(X, E) \xrightarrow{(f, \Phi)} (X', E') \mid f : X \rightarrow X', \Phi \in \mathcal{H}(|X|, \mathcal{E}(E, E'))\}$

$$\bullet \text{ functor } \begin{array}{ccc} \mathbf{Bn}_0(\mathcal{E}, p) & & \\ p_0 \downarrow & & \\ \mathbf{Man}^\rightarrow & & \end{array} : \left\{ \begin{array}{ccc} (X, E) \mapsto & X \times p(E) \xrightarrow{p_1} X & (X, E) \in \text{Ob } \mathbf{Bn}_0(\mathcal{E}, p) \\ & X \times p(E) \xrightarrow{<f \circ p_1, \phi>} X' \times p(E') & \\ (f, \Phi) \mapsto & \begin{array}{ccc} p_1 \downarrow & & \downarrow p_1 \\ X & \xrightarrow{f} & X' \end{array} & (f, \Phi) \in \text{Ar } \mathbf{Bn}_0(\mathcal{E}, p) \end{array} \right.$$

where  $\phi := ev \circ ((p_{E, E'} \circ \Phi) \times 1_{|p(E)|})$ ,  $p_{E, E'} \circ \Phi \in \mathcal{G}(|X|, \mathbf{Man}(p(E), p(E')))$  □

**Definition 5.2.** Category of **locally trivial fibre bundles**  $\mathbf{Bn}(\mathcal{E}, p)$  over site  $(\mathbf{Man}^\rightarrow, \tau_0)$ , where  $\tau_0$  is pullbacks of all open coverings of codomains (i.e., if  $q : Y \rightarrow X \in \text{Ob } \mathbf{Man}^\rightarrow$  and  $\{U_i\}_{i \in I}$  is an open covering of  $X$  then  $\{q^{-1}(U_i) \xrightarrow{q|_{U_i}} U_i\}_{i \in I}$  is a covering of  $q$ ), is a manifold structure of type  $\mathbf{Bn}_0(\mathcal{E}, p)$  over  $\mathbf{Man}^\rightarrow$ . □

A usual way of construction of new fibre bundles from old ones is by fibrewise operations. Let  $\mathcal{E} \xrightarrow{p} \mathbf{Man} \xrightarrow{\pi} \mathbf{Top}$  and  $\mathcal{E}' \xrightarrow{p'} \mathbf{Man} \xrightarrow{\pi} \mathbf{Top}$  be two sequences generating categories of fibre bundles  $\mathbf{Bn}(\mathcal{E}, p)$  and  $\mathbf{Bn}(\mathcal{E}', p')$  of types  $\mathcal{E}$  and  $\mathcal{E}'$ , respectively,  $F : \mathcal{E} \rightarrow \mathcal{E}'$  be an **AMan**-functor. Then there exists a corresponding functor  $\mathbf{Bn}(F) : \mathbf{Bn}(\mathcal{E}, p) \rightarrow \mathbf{Bn}(\mathcal{E}', p')$ .

Denote by **AMan-CAT** an 1-category such that

$$\begin{cases} \text{Ob}(\mathbf{AMan-CAT}) \ni (\mathcal{E}, p), \text{ if } \mathcal{E} \text{ is } \mathbf{AMan}\text{-category, } p : \mathcal{E} \rightarrow \mathbf{Man} \text{ is } \mathbf{AMan}\text{-functor} \\ \text{Ar}(\mathbf{AMan-CAT}) \ni (F : (\mathcal{E}, p) \rightarrow (\mathcal{E}', p')), \text{ if } F : \mathcal{E} \rightarrow \mathcal{E}' \text{ is } \mathbf{AMan}\text{-functors} \end{cases}$$

and by  $\mathbf{Bn}_0$  and  $\mathbf{Bn}$  subcategories of **1-CAT** consisting of categories of trivial and locally trivial fibre bundles with fibres of a fixed type (i.e., of categories like  $\mathbf{Bn}_0(\mathcal{E}, p)$  and  $\mathbf{Bn}(\mathcal{E}, p)$ ) and functors preserving atlases as arrows (see 2.5, remarks). Of course,  $\mathbf{Bn}_0(\mathcal{E}, p) \hookrightarrow \mathbf{Bn}(\mathcal{E}, p)$ .

**Proposition 5.1.** *There are functors  $\mathbf{Bn}_0(-) : \mathbf{AMan-CAT} \rightarrow \mathbf{Bn}_0 \hookrightarrow \mathbf{1-CAT}$  :*

$$\begin{cases} (\mathcal{E}, p) \mapsto \mathbf{Bn}_0(\mathcal{E}, p) & \text{on objects} \\ (F : \mathcal{E} \rightarrow \mathcal{E}') \mapsto \mathbf{Bn}_0(F) : \begin{cases} (X, E) \mapsto (X, F(E)) & (X, E) \in \text{Ob}(\mathbf{Bn}_0(\mathcal{E}, p)) \\ (f, \Phi) \mapsto (f, F_{E, E'} \circ \Phi) & (f, \Phi) \in \text{Ar}(\mathbf{Bn}_0(\mathcal{E}, p)) \end{cases} & \text{on arrows} \end{cases}$$

and  $\mathbf{Bn}(-) : A\mathbf{Man}\text{-}\mathbf{CAT} \rightarrow \mathbf{Bn} \hookrightarrow 1\text{-}\mathbf{CAT}$ , such that  $\mathbf{Bn}(-) = \mathbf{Man}(\mathbf{Bn}_0(-))$  (see 6, remarks) (i.e., to each fibrewise functor there corresponds an actual functor on fibre bundles).

*Proof.* The given assignment for  $\mathbf{Bn}_0(-)$  is obviously functorial. If  $\mathcal{A} := \left\{ \begin{array}{c} U_i \times p(E_i) \\ \downarrow \\ U_i \end{array} \right\}_{i \in I}$  is

an  $\mathcal{E}$ -atlas for  $\begin{array}{c} X \\ \downarrow \\ B \end{array} \in \text{Ob}(\mathbf{Man}^-)$  then  $\mathcal{A}' := \left\{ \begin{array}{c} U_i \times p'(F(E_i)) \\ \downarrow \\ U_i \end{array} \right\}_{i \in I}$  is an  $\mathcal{E}'$ -atlas for  $\begin{array}{c} X' \\ \downarrow \\ B \end{array} :=$

$\left| \mathbf{Bn}(F) \left( \begin{array}{c} X \\ \downarrow \\ B \end{array}, \mathcal{A} \right) \right| \in \text{Ob}(\mathbf{Man}^-)$  (i.e., essentially,  $\mathcal{A}'$  is a compatible family of arrows, if  $\mathcal{A}$  is

compatible, and can be glued to an arrow  $\begin{array}{c} X' \\ \downarrow \\ B \end{array}$ ). So,  $\mathbf{Bn}_0(F)$  and  $\mathbf{Bn}(F)$  preserve atlases.  $\square$

**Remark.** Similarly, there can be defined fibrewise functors of more than one variables (e.g.,  $\mathbf{Bn}(\mathcal{E}, p) \times \mathbf{Bn}(\mathcal{E}', p') \rightarrow \mathbf{Bn}(\mathcal{E}'', p'')$  induced by  $F : \mathcal{E} \times \mathcal{E}' \rightarrow \mathcal{E}''$ , an  $A\mathbf{Man}$ -functor). In this way usual fibrewise operations like  $\times$ ,  $\oplus$ ,  $\otimes$ , etc., are introduced.  $\square$

## 6. Stacks and construction of general manifolds

Stacks give an example of relative higher order Category Theory.  $n$ -categories form an  $(n+1)$ -category, so that (forgetting set-theoretical difficulties)  $\text{Hom}_{(n+1)\text{-}\mathbf{CAT}}(\mathbf{C}, n\text{-}\mathbf{CAT})$  is an  $n$ -category.

**Definition 6.1.** Let  $(\mathbf{B}, \tau)$  be a site ( $\mathbf{B}$  is 1-category),  $F : \mathbf{B}^{op} \rightarrow n\text{-}\mathbf{CAT}$  be a (weak) functor.

- For a sieve  $i_s : s \hookrightarrow \mathbf{B}(-, B)$  ( $n-1$ )-category  $\mathbf{Desc}(s, F) := \text{Hom}_{(n+1)\text{-}\mathbf{CAT}}(s, F)$  is called **descent data** for functor  $F$  and sieve  $s$ .
- $\forall B \in \text{Ob}(\mathbf{B})$  and  $\forall$  sieve  $i_s : s \hookrightarrow \mathbf{B}(-, B)$  there is an induced (**restriction**) functor  $i_s^* : \text{Hom}_{(n+1)\text{-}\mathbf{CAT}}(\mathbf{B}(-, B), F) \rightarrow \text{Hom}_{(n+1)\text{-}\mathbf{CAT}}(s, F)$ . If  $i_s^*$  is full and faithful ( $\forall B$  and  $\forall s$ ) then  $F$  is called **prestack**. If, moreover, it is an equivalence  $F$  is called **stack**, i.e.,  $F$  is **prestack** iff  $\forall B, s \text{ Hom}_{(n+1)\text{-}\mathbf{CAT}}(s, F) \xleftarrow[i_s^*, \text{full, faith.}]{i_s^*} \text{Hom}_{(n+1)\text{-}\mathbf{CAT}}(\mathbf{B}(-, B), F) \xrightarrow[\text{Yoneda}]{\sim} F(B)$  and **stack** iff  $i_s^*$  is an equivalence.  $\square$

For  $n=1$  there is another definition of stack via matching families [Moe, Vis].

Denote by  $\text{PreSt}(\mathbf{B}^{op}, n\text{-}\mathbf{CAT})$ ,  $\text{St}(\mathbf{B}^{op}, n\text{-}\mathbf{CAT}) \hookrightarrow \text{Hom}_{(n+1)\text{-}\mathbf{CAT}}(\mathbf{B}^{op}, n\text{-}\mathbf{CAT})$  full subcategories of prestacks and staks respectively.

**Proposition 6.1.** *Both inclusions*

$$\mathbf{St}(\mathbf{B}^{op}, 1\text{-}\mathbf{CAT}) \xleftarrow[\underset{i_0}{\perp}]{\Phi} \mathbf{PreSt}(\mathbf{B}^{op}, 1\text{-}\mathbf{CAT}) \xleftarrow[\underset{i_1}{\perp}]{\Psi} \mathbf{Hom}_{2\text{-}\mathbf{CAT}}(\mathbf{B}^{op}, 1\text{-}\mathbf{CAT})$$

have left adjoints.

*Proof.* See [Moe, Vis]. □

### Construction of manifolds of type $\mathbf{E}$ over site $(\mathbf{B}, \tau)$

1. Factor (1)-functor  $F : \mathbf{E} \rightarrow (\mathbf{B}, \tau)$  through a free fibration (see **Lemma 1.1**)
 

$$\begin{array}{ccc} \mathbf{E} & \xrightarrow{i} & 1/F \\ & \searrow F & \downarrow \text{dom} \\ & & \mathbf{B} \end{array}$$
  2. For fibration  $\begin{array}{c} 1/F \\ \text{dom} \downarrow \\ \mathbf{B} \end{array}$  form a corresponding (weak) functor  $\hat{F} : \mathbf{B}^{op} \rightarrow 1\text{-}\mathbf{CAT}$  and complete it to a stack  $(\Phi \circ \Psi)\hat{F} : \mathbf{B}^{op} \rightarrow 1\text{-}\mathbf{CAT}$  with respect to topology  $\tau$ .
  3. Get back (by Grothendieck construction) from stack  $(\Phi \circ \Psi)\hat{F} : \mathbf{B}^{op} \rightarrow 1\text{-}\mathbf{CAT}$  to a fibration
 

$$\begin{array}{ccc} \tilde{\mathbf{E}} & \xleftarrow{\quad} & 1/F \\ p \downarrow & \swarrow \text{dom} & \\ \mathbf{B} & & \end{array}$$
  4. Choose a (correct) class of arrows  $\mathcal{M}$  in  $\mathbf{B}$  representing 'embeddings of simple pieces into manifolds'.
  5. Take a full subcategory  $\mathbf{E}\text{-}\mathbf{Man} \hookrightarrow \tilde{\mathbf{E}}$  consisting of all objects  $(B, \mathcal{E}) \in \text{Ob}(\tilde{\mathbf{E}})$  such that  $\exists$  a sieve  $s \hookrightarrow \mathbf{B}(-, B)$  (depending on  $B$ ) and  $\forall f \in s \ \mathbf{Cart}_f(\mathcal{E}) = ((\Phi \circ \Psi)\hat{F}(f))(\mathcal{E}) : (df \rightarrow F(E)) \in \mathcal{M}$  for some  $E \in \text{Ob}(\mathbf{E})$ . Then
 

$$\begin{array}{ccccc} \mathbf{E}\text{-}\mathbf{Man} & \hookrightarrow & \tilde{\mathbf{E}} & \xleftarrow{\quad} & 1/F \\ & \searrow p_F & p \downarrow & \swarrow \text{dom} & \\ & & \mathbf{B} & & \end{array}$$

 is the required
- category of manifolds of type  $\mathbf{E}$  over base site  $(\mathbf{B}, \tau)$ . □

#### Remarks.

- Depending on the choice of class  $\mathcal{M}$  categories  $\mathbf{E}\text{-}\mathbf{Man}$  will be different (so,  $\mathcal{M}$  is an additional parameter). For cases of usual manifolds (smooth real or complex)  $\mathcal{M}$  is always class of topological embeddings of open subspaces.
- An object  $(B, \mathcal{E})$  in  $\mathbf{E}\text{-}\mathbf{Man}$  consists of a base object  $B$  and an 'atlas'  $\mathcal{E}$ , where  $\mathcal{E}$  is a class of compatible charts  $(U \rightarrow F(E)) \in \mathcal{M}$ ,  $U \in \text{Ob}(\text{Im} F)$ ,  $E \in \mathbf{E}$ . All arrows are represented by vertical arrows for the stack completion of  $1/F$ .
- $\text{Im}(p_F) \supset \text{Im}(F)$ .

$\mathbf{E}\text{-}\mathbf{Man}$

$\downarrow p_F$   
 $\mathbf{B}$

- The resulting category of manifolds is not usually a fibration.
- Denote by  $\mathbf{Man}_0 \hookrightarrow 1\text{-}\mathbf{CAT}$  a category consisting of subcategory of  $\mathbf{E}\text{-}\mathbf{Man}$  of trivial manifolds of type  $\mathbf{E}$  for each type  $\mathbf{E}$  and functors 'mapping  $\mathbf{E}$ -atlases to  $\mathbf{E}'$ -atlases'. Respectively, by  $\mathbf{Man} \hookrightarrow 1\text{-}\mathbf{CAT}$  a category consisting of  $\mathbf{E}\text{-}\mathbf{Man}$  for each type  $\mathbf{E}$  and functors 'mapping

**E**-atlases to **E'**-atlases'. Then  $\exists$  'manifoldification' functor  $\mathbf{Man}(-) : \mathbf{Man}_0 \rightarrow \mathbf{Man}$ . Inclusion functor  $Ob(\mathbf{Man}_0) \ni \mathbf{E-Man}_0 \hookrightarrow \mathbf{E-Man} \in Ob(\mathbf{Man})$  is natural in **E** (and itself is preserving atlases).  $\square$

**Example** (single manifold)

Let **E** =category consisting of  $k^n, n = 0, 1, \dots, (k = \mathbb{R} \text{ or } \mathbb{C})$  and all smooth (analytic) local isomorphisms,  $\mathcal{M}$  be embeddings of open subspaces, **B** =union of category of open subsets of a space  $X$  with inclusion arrows, category **E**, and arrows from  $\mathcal{M}$  with codomains in **E**,  $F : \mathbf{E} \rightarrow \mathbf{B}$  be the inclusion functor. Assignment  $X \subset U \mapsto \{f : U \rightarrow k^n \mid n = 0, 1, \dots, f \in \mathcal{M}\}$  gives a prestack on  $X$ . It is a nontrivial stack iff  $X$  is a manifold.  $(X, \mathcal{E}) \in Ob(\mathbf{E-Man})$  iff  $\mathcal{E}$  is an atlas on  $X$ .

## 7. Lifting problem for a group action

Let **Grp** be a category of groups,  $(-)\mathbf{Grp} : 1\text{-CAT} \rightarrow 1\text{-CAT}$  be a functor which assigns (in an obvious way) to each category without group action a category with groups actions, namely,

$$\begin{cases} \mathcal{C} \mapsto \mathbf{Grp-C} & \mathcal{C} \in Ob(1\text{-CAT}) \\ (F : \mathcal{C} \rightarrow \mathcal{C}') \mapsto (\mathbf{Grp-F} : \mathbf{Grp-C} \rightarrow \mathbf{Grp-C}') & F \in Ar(1\text{-CAT}) \end{cases} \quad \text{where:}$$

**Grp-C** consists of triples  $(G, C, \rho)$  ( $G \in Ob(\mathbf{Grp}), C \in Ob(\mathcal{C}), \rho : G \rightarrow \mathbf{Aut}(C)$  is a group homomorphism) as objects, and pairs  $(\sigma : G \rightarrow G', f : C \rightarrow C') : (G, C, \rho) \rightarrow (G', C', \rho')$  (s.t.  $\forall g \in G \quad \rho'(\sigma(g)) \circ f = f \circ \rho(g)$ ) as arrows,

$$\mathbf{Grp-F} : \begin{cases} (G, C, \rho) \mapsto (G, F(C), F_{C,C} \circ \rho) & (G, C, \rho) \in Ob(\mathbf{Grp-C}) \\ (\sigma, f) \mapsto (\sigma, F(f)) & (\sigma, f) \in Ar(\mathbf{Grp-C}) \end{cases}$$

$[(\sigma, F(f))$  is an (equivariant) arrow in **Grp-C'** because  $F(\rho'(\sigma(g))) \circ F(f) = F(f) \circ F(\rho(g)) \quad \forall g \in G]$

**Proposition 7.1.** If  $\begin{array}{c} \mathbf{E} \\ \downarrow p \\ \mathbf{B} \end{array}$  is a structure over **B** (i.e., all isomorphisms of type  $(B' \xrightarrow{\sim} p(E)) \in$

$Ar \mathbf{B}$  can be lifted to isomorphisms  $(E' \xrightarrow{\sim} E) \in Ar \mathbf{E}$ ) then  $\begin{array}{c} \mathbf{Grp-E} \\ \downarrow \mathbf{Grp-p} \\ \mathbf{Grp-B} \end{array}$  is a structure over

**Grp-B.**

*Proof.* If  $(\varphi, f) : (G', B', \rho') \xrightarrow{\sim} (G, p(E), p \circ \rho)$  is an iso then  $\exists \begin{pmatrix} E' \\ B' \end{pmatrix} \xrightarrow[\hat{f}]{\sim} \begin{pmatrix} E \\ B \end{pmatrix}$ , iso, because  $\begin{array}{c} \mathbf{E} \\ \downarrow p \\ \mathbf{B} \end{array}$

is a structure over **B**. Regard the diagram (of group homomorphisms)

$$\begin{array}{ccccc} \mathbf{Aut}_{\mathbf{E}}(E') & \xleftarrow{\hat{f}^{-1} \circ_{\mathbf{E}} - \circ_{\mathbf{E}} \hat{f}} & \mathbf{Aut}_{\mathbf{E}}(E) & & \\ \downarrow p & \swarrow \rho'' & \nearrow \rho & & \downarrow p \\ & G' & \xrightarrow[\sim]{\varphi} & G & \\ & \swarrow \rho' & \searrow p \circ \rho & & \\ \mathbf{Aut}_{\mathbf{B}}(B') & \xleftarrow{f^{-1} \circ_{\mathbf{B}} - \circ_{\mathbf{B}} f} & \mathbf{Aut}_{\mathbf{B}}(p(E)) & & \end{array}$$

(\*\*)  $\quad (*)$

$$\rho''(g') := \hat{f}^{-1} \circ_{\mathbf{E}} \rho(\varphi(g')) \circ_{\mathbf{E}} \hat{f}$$

(\*) commutes because  $f \circ_{\mathbf{B}} \rho'(g') = (p \circ \rho)(\varphi(g')) \circ_{\mathbf{B}} f$  by equivariance condition.

(\*\*) commutes because  $p(\rho''(g')) = f^{-1} \circ_{\mathbf{B}} p(\rho(\varphi(g'))) \circ_{\mathbf{B}} f = \rho'(g')$ .

So,  $\exists$  iso  $\left( \begin{smallmatrix} (G', E', \rho'') \\ (G', B', \rho') \end{smallmatrix} \right) \xrightarrow[\sim]{((\varphi, f))} \left( \begin{smallmatrix} G, E, \rho \\ G, B, p \circ \rho \end{smallmatrix} \right)$ , i.e.  $\text{Grp-}\mathbf{E} \downarrow \text{Grp-}p \text{Grp-}\mathbf{B}$  is a structure over **Grp-B**.  $\square$

There is a commutative diagram in **1-CAT**  $\begin{array}{ccc} \mathbf{E} & \longleftarrow & \text{Grp-}\mathbf{E} \\ p \downarrow & & \downarrow \text{Grp-}p \\ \mathbf{B} & \longleftarrow & \text{Grp-}\mathbf{B} \end{array}$  (where horizontal arrows forget group actions). So, there exists a forgetful fiber functor  $\mathbf{E}_B \leftarrow \text{Grp-}\mathbf{E}_{(G,B,\rho)}$ .

**Definition 7.1.** For a given  $G$ -action  $(G, B, \rho) \in \text{Ob}(\text{Grp-}\mathbf{B}_B)$ , an object  $E \in \text{Ob}(\mathbf{E}_B)$

admits lifting of  $G$ -action if  $\exists (G, E, \rho') \in \text{Ob}(\text{Grp-}\mathbf{E}_{(G,B,\rho)})$ , i.e.,  $\begin{array}{ccc} E & \longleftarrow & (G, E, \rho') \\ p \downarrow & & \downarrow \text{Grp-}p \\ B & \longleftarrow & (G, B, \rho) \end{array}$  (essentially,  $\rho = p \circ \rho'$ ).  $\square$

**Lifting problem** for a  $G$ -action  $\rho : G \rightarrow \mathbf{Aut}_{\mathbf{B}}(B)$  is equivalent to completion of the diagram of group homomorphisms with exact row  $1 \longrightarrow \mathbf{Aut}_{\mathbf{E}_B}(E) \longrightarrow \mathbf{Aut}_{\mathbf{E}}(E) \xrightarrow{p} \mathbf{Aut}_{\mathbf{B}}(B)$

$$\begin{array}{ccc} & \uparrow & \\ & \rho' \mid ? & \\ & \downarrow & \\ G & \nearrow \rho & \end{array}$$

where  $\mathbf{Aut}_{\mathbf{E}_B}(E)$  are vertical automorphisms of  $E$  over  $B$ .

For single element  $g \in \mathbf{Aut}_{\mathbf{B}}(B)$  there is a simple criterion of existence of  $g' \in \mathbf{Aut}_{\mathbf{E}}(E)$  such that  $p(g') = g$  (see [Kom]).

**Proposition 7.2.** For a fibration  $\begin{array}{c} \mathbf{E} \\ p \downarrow \\ \mathbf{B} \end{array}$  (or structure over  $\mathbf{B}$ )  $g \in \mathbf{Aut}_{\mathbf{B}}(B)$  can be lifted to  $g' \in \mathbf{Aut}_{\mathbf{E}}(E)$  iff  $\mathbf{Cart}_g(E) \xrightarrow{\sim} E$  (vertical iso).

*Proof.* ' $\iff$ '

$$\begin{array}{ccc} E & & \\ \searrow \text{vertical} & \nearrow g' & \\ \mathbf{Cart}_g(E) & \xrightarrow[\sim]{\tilde{g}} & E \end{array}$$

$$B \xrightarrow[\sim]{g} B$$

$\square$

**Proposition 7.3.** If  $\begin{array}{c} \mathbf{E} \\ \downarrow \\ \mathbf{B} \end{array}$  is a structure on  $\mathbf{B}$  (or (co)fibration with **unique** (co)Cartesian lifting),

and  $(G, B, \rho) \in \text{Ob}(\mathbf{Grp}\text{-}\mathbf{B})$ , then  $\exists$  a representation

$$\begin{array}{ccc} G & \xrightarrow{\mathcal{R}} & \mathbf{Aut}_{1\text{-CAT}}(\mathbf{E}_B) \\ & \searrow \rho & \uparrow \\ & & \mathbf{Aut}_B(B) \end{array}$$

$$\text{where } \mathcal{R}(g) : \begin{cases} E \mapsto \mathbf{Cart}_{\rho(g)}(E) & E \in \text{Ob}(\mathbf{E}_B) \\ \begin{array}{ccc} E & \xleftarrow{\widetilde{\rho(g)}} & \mathbf{Cart}_{\rho(g)}(E) \\ f \downarrow & \mapsto & \downarrow \mathcal{R}(g)(f) \\ E' & \xleftarrow{\widetilde{\rho(g)}} & \mathbf{Cart}_{\rho(g)}(E') \end{array} & f \in \text{Ar}(\mathbf{E}_B) \end{cases}$$

*Proof* is straightforward.  $\square$

**Corollary.** If  $E \in \text{Ob}(\mathbf{E}_B)$  is such that  $\forall g \in G \mathbf{Cart}_{\rho(g)}(E) = E$  then Cartesian lifting  $\rho(g) \mapsto \widetilde{\rho(g)}$  lifts action  $(G, B, \rho) \in \text{Ob}(\mathbf{Grp}\text{-}\mathbf{B})$  to the action  $(G, E, \tilde{\rho}) \in \text{Ob}(\mathbf{Grp}\text{-}\mathbf{E}_{(G, B, \rho)})$ .  $\square$

**Example** (Covering Space)

A covering space is a (co)fibration  $\begin{array}{c} E \\ p \downarrow \\ B \end{array}$  over groupoid  $B$  with unique (co)cartesian lifting in

which all morphisms are (co)cartesian. Moreover, representation  $\mathbf{Aut}(b) \rightarrow \mathbf{Aut}(E_b)$ ,  $b \in \text{Ob}(B)$  (induced by (co)cartesian lifting) is transitive on objects of  $E_b$ .

**Proposition 7.4.** For a covering space  $\begin{array}{c} E \\ p \downarrow \\ B \end{array}$  over connected groupoid  $B$

- $\mathbf{Aut}\left(\begin{array}{c} E \\ p \downarrow \\ B \end{array}\right) \simeq \mathbf{Aut}(E_b)$  (where  $g \in \mathbf{Aut}(E_b)$  iff  $g \circ f^* = f^* \circ g$ ,  $f^* \equiv \text{coCart}_f$ ,  $\forall f \in \mathbf{Aut}(b)$ ),
- $\mathbf{Aut}(E_b) \simeq W(\mathbf{Stab}(e)) \simeq N(\mathbf{Stab}(e))/\mathbf{Stab}(e)$  (where  $\mathbf{Stab}(e) \hookrightarrow \mathbf{Aut}(b)$  is the stabilizer of an object  $e \in \text{Ob}(E_b)$ ,  $N(\mathbf{Stab}(e))$ ,  $W(\mathbf{Stab}(e))$  are its normalizer and Weil group respectively).

*Proof.*

- An automorphism  $g$  of covering space  $p$  is given by family of fiberwise functors  $g_b$ ,  $b \in \text{Ob}(B)$ , such that  $f^* \circ g_b = g_{b'} \circ f^*$ ,  $f^* \equiv \text{coCart}_f$ ,  $\forall (f : b \rightarrow b') \in \text{Ar}(B)$ . Take  $g_b \in \mathbf{Aut}(E_b)$  and define  $g_{b'} := h^* \circ g_b \circ (h^*)^{-1}$  for some  $h : b \rightarrow b'$ . Then  $g_{b'}$  is well-defined (if  $h_1 : b \rightarrow b'$  is another morphism then  $h^* \circ g_b \circ (h^*)^{-1} = h_1^* \circ g_b \circ (h_1^*)^{-1}$  since  $(h_1^{-1} \circ h)^* \circ g_b = (h_1^*)^{-1} \circ h^* \circ g_b = g_b \circ (h_1^*)^{-1} \circ h^* = g_b \circ (h_1^{-1} \circ h)^* = g_b \circ (h_1^{-1} \circ h)^*$ ,  $h_1^{-1} \circ h \in \mathbf{Aut}(b)$ ), and it is an automorphism of covering space  $p$  (if  $f : b' \rightarrow b''$  then  $f^* \circ g_{b'} = g_{b''} \circ f^*$  since  $f^* \circ h^* \circ g_b = g_{b''} \circ f^* \circ h^*$ ,  $f \circ h : b \rightarrow b''$ ).
- See [May].  $\square$

### 7.1. Lifting of a groupoid action for a sheaf.

**Definition 7.1.1.** Let  $(\mathbf{Top}, \tau_0)$  be a site for all open coverings on topological spaces.

- **Set-valued presheaf**  $P : \mathbf{Top}^{op} \rightarrow \mathbf{Set}$  is a **sheaf** iff  $\forall$  sieve  $S \hookrightarrow \mathbf{B}(-, B)$  and  $\forall$  natural transformation  $f : S \rightarrow P \exists! \hat{f} : \mathbf{B}(-, B) \rightarrow P$  such that

$$\begin{array}{ccc} S & \hookrightarrow & \mathbf{B}(-, B) \\ & \searrow \forall f & \downarrow \exists! \hat{f} \\ & & P \end{array}$$

- **Cat-valued presheaf**  $P : \mathbf{Top}^{op} \rightarrow \mathbf{Cat}$  is a **sheaf** iff its object and morphism parts are sheaves, i.e.,  $\mathbf{Top}^{op} \xrightarrow{P} \mathbf{Cat} \xrightarrow{Ob} \mathbf{Set}$  and  $\mathbf{Top}^{op} \xrightarrow{P} \mathbf{Cat} \xrightarrow{Mor} \mathbf{Set}$  are **Set-valued sheaves**.
- For presheaf  $P : \mathbf{Top}^{op} \rightarrow \mathbf{Cat}$ , space  $X \in Ob(\mathbf{Top})$  and sieve  $S \hookrightarrow \mathbf{Top}(-, X)$  matching family of objects  $\tilde{E} : S \rightarrow Ob \circ P$  (nat. trans.) (or matching family of arrows  $\tilde{f} : S \rightarrow Mor \circ P$  (nat. trans.)) has a **germ**  $\mathbf{germ}_x(\tilde{E})$  (respectively,  $\mathbf{germ}_x(\tilde{f})$ ) at point  $x \in X$  iff  $\exists Colim_{s \in S, Im(s) \ni x}(\tilde{E}(s)) =: \mathbf{germ}_x(\tilde{E})$  (respectively,  $Colim_{s \in S, Im(s) \ni x}(\tilde{f}(s)) =: \mathbf{germ}_x(\tilde{f})$ ) (if germ exists it is unique up to iso and does not depend on the choice of sieve  $S$ ).
- **Etale space** is  $E := \coprod_{x \in X} \mathbf{germ}_x(\tilde{E})$  (respectively,  $f := \coprod_{x \in X} \mathbf{germ}_x(\tilde{f})$ ) (depending on two

variables: 'point'  $x \in X$  and 'matching family'  $\tilde{E}$  or  $\tilde{f}$ ) with topology generated by basic open sets  $(U, \{\mathbf{germ}_x(\tilde{E}) \mid x \in U\})$  (or,  $(U, \{\mathbf{germ}_x(\tilde{f}) \mid x \in U\})$ ),  $U$  is open in  $X$ . There is a natural continuous projection  $p : E \rightarrow X : (x, \mathbf{germ}_x(\tilde{E})) \mapsto x$  (respectively,  $p : f \rightarrow X : (x, \mathbf{germ}_x(\tilde{f})) \mapsto x$ ) which is a local homeomorphism.  $\square$

**Lemma 7.1.1.** Every fibration is a cofibration with respect to iso's (every cofibration is a fibration with respect to iso's).

*Proof.* Let  $\begin{array}{c} \mathbf{E} \\ p \downarrow \\ \mathbf{B} \end{array}$  be a fibration, and  $p(E) \xrightarrow{f} B'$  be an iso in  $\mathbf{B}$ . Then  $\tilde{f} := (\widetilde{f^{-1}})^{-1} : E \rightarrow E'$

(where  $\sim$  on the right is a cartesian lifting) is a cocartesian lifting of  $f$  (obvious).  $\square$

**Corollary.** For a (co)fibration  $\begin{array}{c} \mathbf{E} \\ p \downarrow \\ \mathbf{B} \end{array}$  for each iso  $(f : B \xrightarrow{\sim} B') \in Ar \mathbf{B}$  inverse image  $\mathbf{Cart}_f :$

$\mathbf{E}_{B'} \rightarrow \mathbf{E}_B : s_{B'} \mapsto f^*(s_{B'})$  and direct image  $\mathbf{Cart}_f : \mathbf{E}_B \rightarrow \mathbf{E}_{B'} : s_B \mapsto f_*(s_B)$  (where  $s_B$  is a 'section' (object or morphism) over  $B$ ) exist.  $\square$

**Definition 7.1.2.**

- For a space  $X \in Ob(\mathbf{Top})$  **groupoid of local homeomorphisms** of  $X$  is a subcategory  $\mathbf{Gr}_X \hookrightarrow \mathbf{Top}$  such that  $\begin{cases} Ob(\mathbf{Gr}_X) & \text{are open subsets in } X \\ Ar(\mathbf{Gr}_X) & \text{are iso's in } \mathbf{Top} \text{ (between open subsets in } X) \end{cases}$   
(Nonfull) subcategory  $\mathbf{Gr}_{X,x} \hookrightarrow \mathbf{Gr}_X$  with objects  $U \ni x$  and morphisms  $f : U \rightarrow V, f(x) = x$ , is called **groupoid of local homeomorphisms of  $X$  with fixed point  $x \in X$** .
- $X$  is **transitive** with respect to  $\mathbf{Gr}_X$  if  $\forall x, y \in X \exists U, V \in Ob(\mathbf{Gr}_X), (f : U \rightarrow V) \in Ar(\mathbf{Gr}_X)$  such that  $U \ni x, V \ni y, f(x) = y$ .

- For a (co)fibration  $\begin{array}{c} \mathbf{E} \\ p \downarrow \\ \mathbf{Top} \end{array}$  with unique (co)cartesian ligting and space  $X \in Ob(\mathbf{Top})$  two actions

of  $\mathbf{Gr}_X$  on local sections over  $X$  are defined:

**left action**  $\forall f \in \mathbf{Gr}_X(U, V) \quad f^* \equiv \mathbf{Cart}_f : \mathbf{E}_V \rightarrow \mathbf{E}_U : s_V \mapsto f^* s_V$

**right action**  $\forall f \in \mathbf{Gr}_X(U, V) \quad f_* \equiv \mathbf{coCart}_f : \mathbf{E}_U \rightarrow \mathbf{E}_V : s_U \mapsto f_* s_U$

(where  $s_V, s_U$  are objects or morphisms).

- To each of actions  $f^*, f_*$  (on local sections of  $\mathbf{E}_X$ ) there correspond respectively left and right actions of  $\mathbf{Gr}_{X,x}$  on  $\{\mathbf{germ}_x(\tilde{s}) \mid \tilde{s} \text{ is a matching family of local sections of } \mathbf{E}_X\}$ . If  $\mathfrak{s} = \mathbf{germ}_x(s_U)$  is a germ at point  $x$  presented by a local section  $s_U$  (i.e.,  $\mathfrak{s} = \mathbf{Colim}_{U \supset V \ni x} (s_U|_V)$ )

then if  $(f : W \rightarrow V) \in \mathbf{Gr}_{X,x}(W, V)$

**left action**  $f^* \mathfrak{s} := \mathbf{germ}_x((f|_{f^{-1}(U \cap V)})^*(s_U|_{U \cap V}))$

**right action**  $f_* \mathfrak{s} := \mathbf{germ}_x((f|_{U \cap W})_*(s_U|_{U \cap W}))$ .

- For a subgroupoid  $\mathbf{G} \hookrightarrow \mathbf{Gr}_X$ ,  $\begin{pmatrix} s \\ X \end{pmatrix} \in \begin{pmatrix} \mathbf{E} \\ p \downarrow \\ \mathbf{Top} \end{pmatrix}$  is **G-invariant** if  $\forall (f : U \rightarrow V) \in \mathbf{Ar}(\mathbf{G})$

$f^*(s_V) = s_U$  (or  $f_*(s_U) = s_V$ ) (where:  $s_U := s|_U := (i : U \hookrightarrow X)^* s$ ,  $\begin{pmatrix} \mathbf{E} \\ p \downarrow \\ \mathbf{B} \end{pmatrix}$  is a fibration with

unique cartesian lifting (or a local structure with respect to inclusions of open sets),  $s$  is a section (object or morphism) over  $X$ ). In other words, **G**-invariant sections **admit lifting** of groupoid **G**.

- $\mathbf{germ}_{x,y}(f)$  of a map  $(f : U \rightarrow V) \in \mathbf{Ar}(\mathbf{Gr}_X)$ , such that  $f(x) = y$ , is an equivalence class of maps  $\{g \in \mathbf{Ar}(\mathbf{Gr}_X) \mid g(x) = y, \exists \text{ opens } W_x \ni x, W_y \ni y, \text{ such that } \exists \text{ the same restrictions } f|_{W_x W_y} = g|_{W_x W_y} \in \mathbf{Ar}(\mathbf{Gr}_X)\}$ . Assume,  $\mathfrak{s}_x = \mathbf{germ}_x(s_{U_1})$ ,  $\mathfrak{s}'_y = \mathbf{germ}_y(s'_{V_1})$ . Then

**left action**  $(\mathbf{germ}_{x,y}(f))^* \mathfrak{s}'_y := \mathbf{germ}_x((f|_{f^{-1}(V \cap V_1)})^*(s'_{V_1}|_{V \cap V_1}))$

**right action**  $(\mathbf{germ}_{x,y}(f))_* \mathfrak{s}_x := \mathbf{germ}_y((f|_{(U \cap U_1)})_*(s_{U_1}|_{U \cap U_1}))$ . □

### Lemma 7.1.2.

- Let  $X$  be a topological space,  $\mathbf{G} \hookrightarrow \mathbf{Gr}_X$  be a subgroupoid,  $\mathbf{G}_x \hookrightarrow \mathbf{G}$  be a subgroupoid of pointed maps with fixed point  $x \in X$ . Then  $\forall x, y \in X$  and  $\forall f \in \mathbf{Ar}(\mathbf{G})$ , s.t.  $f(x) = y$ ,  $\mathbf{germ}_{x,x}(\mathbf{G}_x) = \mathbf{germ}_{y,x}(f^{-1}) \cdot \mathbf{germ}_{y,y}(\mathbf{G}_y) \cdot \mathbf{germ}_{x,y}(f)$  (for certain unique composite  $\cdot$  of germs of maps).

- If  $\mathfrak{s}_x = \mathbf{germ}_x(s_U) \in S_x \subset S$  is a point of etale space  $\begin{pmatrix} S \\ \downarrow \\ X \end{pmatrix}$  (corresponding to objects or mor-

phisms over  $X$  for a fibration  $\begin{pmatrix} \mathbf{E} \\ p \downarrow \\ \mathbf{Top} \end{pmatrix}$ ) then  $\mathfrak{s}_x$  is **G**<sub>x</sub>-invariant iff it is  $\mathbf{germ}_{x,x}(\mathbf{G}_x)$ -invariant.

- If **G** is transitive on  $X$  and  $\mathfrak{s}_x$  is **G**<sub>x</sub>-invariant then  $\forall f, g \in \mathbf{Ar}(\mathbf{G})$ , s.t.  $f(x) = y, g(x) = y$ , there is a unique induced germ at point  $y$   $(\mathbf{germ}_{x,y}(f))_* \mathfrak{s}_x = (\mathbf{germ}_{x,y}(g))_* \mathfrak{s}_x$  and this germ



is  $\mathbf{G}_y$ -invariant (or, respectively,  $(\mathbf{germ}_{y,x}(f^{-1}))^* \mathfrak{s}_x = (\mathbf{germ}_{y,x}(g^{-1}))^* \mathfrak{s}_x$  is a unique  $\mathbf{G}_y$ -invariant germ at point  $y$ ), i.e.  $\mathfrak{s}_x$  can be distributed in a unique way over all  $X$  (to give rise

a section  $s : X \rightarrow S$  of etale space  $\begin{array}{c} S \\ \downarrow \\ X \end{array}$  consisting of invariant germs at each point).  $\square$

**Proposition 7.1.1.** For sheaf  $P : \mathbf{Top}^{op} \rightarrow \mathbf{CAT}$ , space  $X \in \mathbf{Ob}(\mathbf{Top})$ , and transitive groupoid  $\mathbf{G} \hookrightarrow \mathbf{Gr}_X$   $\mathbf{G}$ -invariant sections over  $X$  are in bijective correspondence with a subset of  $\mathbf{G}_x$ -invariant germs (of local sections) for a fixed point  $x \in X$ .

*Proof.* To each  $\mathbf{G}$ -invariant section over  $X$  there corresponds  $\mathbf{G}_x$ -invariant germ of this section at point  $x$ . Conversely, by lemma 7.1.2, each  $\mathbf{G}_x$ -invariant germ generates a section of the corresponding etale space. When this section is continuous there is a global section over  $X$  of sheaf  $P$  (which is locally invariant).  $\square$

**Remark.** B.P. Komrakov [Kom] asserts (without a proof) that the above bijective correspondence is with the whole set of  $\mathbf{G}_x$ -invariant germs. But, without additional assumptions it is not clear why the corresponding section of invariant germs is continuous and the sheaf section is invariant.  $\square$

## 8. Equivalence, groups, actions

Let  $\mathcal{R}$  be a category of sets with a given equivalence relation for each set. There are following functors

- **forgetful**  $p : \mathcal{R} \rightarrow \mathbf{Set} : (A, R) \mapsto A$
- **quotient**  $Q : \mathcal{R} \rightarrow \mathbf{Set} : (A, R) \mapsto A/R$
- **inclusion**  $\Delta : \mathbf{Set} \rightarrow \mathcal{R} : A \mapsto (A, \Delta_A), \Delta_A := \{(a, a) \mid a \in A\}$

such that  $\mathcal{R} \begin{array}{c} \xrightarrow{Q} \\ \perp \\ \xleftarrow{\Delta} \end{array} \mathbf{Set}$ , i.e.  $\mathbf{Set}(Q(A, R), B) \xrightarrow[\text{nat.iso}]{} \mathcal{R}((A, R), \Delta(B)) : f \mapsto f \circ \pi$ , where

$A \xrightarrow{\pi} A/R$  is the canonical projection (so, quotient object  $Q(A, R)$  represents functor  $\mathcal{R}((A, R), \Delta(-)) : \mathbf{Set} \rightarrow \mathbf{Set}$ ).

For arbitrary category  $\mathbf{C}$  equivalence relation on objects is introduced as usual via hom-sets.

**Definition 8.1.**

- A functor  $R : \mathbf{C}^{op} \rightarrow \mathcal{R}$  is called an **equivalence relation on object**  $C \in \mathbf{Ob} \mathbf{C}$  iff

$$\begin{array}{ccc} \mathbf{C}^{op} & \xrightarrow{R} & \mathcal{R} \\ & \searrow & \downarrow p \\ \mathbf{C}(-, C) & & \mathbf{Set} \end{array} \quad (\text{i.e. usual equivalence relations are introduced on hom-sets } \mathbf{C}(C', C), C' \in$$

$\mathbf{Ob} \mathbf{C}$  and they are preserved under precomposition  $- \circ f, f : C'' \rightarrow C'$ ).

- Let  $\mathbf{C}_{\mathcal{R}}$  be a category such that

$\mathbf{Ob}(\mathbf{C}_{\mathcal{R}})$  are pairs  $(C, R)$ ,  $C \in \mathbf{Ob} \mathbf{C}$ ,  $R$  is an equivalence relation on  $C$ ,

$\mathbf{Ar}(\mathbf{C}_{\mathcal{R}})$  are maps  $(C, R) \xrightarrow{(f, F)} (C', R')$ , where  $(f : C \rightarrow C') \in \mathbf{Ar} \mathbf{C}$  and  $F : R \Rightarrow R'$  is a

natural transformation of equivalence relations such that

$$\begin{array}{ccc} & \mathbf{C}(-, C) & \\ \mathbf{C}^{op} & \begin{array}{c} \xrightarrow{R} \\ \Downarrow F \\ \xrightarrow{R'} \end{array} & \mathcal{R} \xrightarrow{p} \mathbf{Set} \\ & \mathbf{C}(-, C') & \end{array}$$

$p_F = \mathbf{C}(-, f)$  (it means  $(C, R) \xrightarrow{(f, F)} (C', R')$  is a morphism in  $\mathbf{C}_{\mathcal{R}}$  iff  $f : C \rightarrow C'$  is an arrow in  $\mathbf{C}$  and  $f \circ -$  preserves equivalence relation, i.e. if  $g_1 \sim_R g_2$  then  $f \circ g_1 \sim_{R'} f \circ g_2$  for  $g_1, g_2 : X \rightarrow C$ ).  $\square$

$\mathbf{C}_{\mathcal{R}}$  is an analogue of  $\mathcal{R}$  for arbitrary category  $\mathbf{C}$ . Again, there are the following functors

- **forgetful**  $p : \mathbf{C}_{\mathcal{R}} \rightarrow \mathbf{C} : (C, R) \mapsto C$
- **inclusion**  $\Delta : \mathbf{C} \rightarrow \mathbf{C}_{\mathcal{R}} : C \mapsto (C, \Delta \circ \mathbf{C}(-, C))$ , where  $\Delta : \mathbf{Set} \rightarrow \mathcal{R} : A \mapsto (A, \Delta_A)$ ,  $\Delta_A := \{(a, a) \mid a \in A\}$
- **quotient**  $Q : \mathbf{C}_{\mathcal{R}} \rightarrow \mathbf{C} : (C, R) \mapsto C/R$  which is a left adjoint to  $\Delta : \mathbf{C} \rightarrow \mathbf{C}_{\mathcal{R}}$ , i.e.

$$\mathbf{C}_{\mathcal{R}} \begin{array}{c} \xrightarrow{Q} \\ \perp \\ \xleftarrow{\Delta} \end{array} \mathbf{C} \quad \text{or} \quad \mathbf{C}(Q(C, R), C') \xrightarrow[\text{nat.iso}]{\sim} \mathbf{C}_{\mathcal{R}}((C, R), \Delta(C')) \quad (\text{quotient object } C/R := Q(C, R))$$

represents functor  $\mathbf{C}_{\mathcal{R}}((C, R), \Delta(-)) : \mathbf{C} \rightarrow \mathbf{Set}$  which means that  $\exists$  an arrow  $\pi : (C, R) \rightarrow \Delta(Q(C, R))$  such that  $\forall f : (C, R) \rightarrow \Delta(C') \exists! \hat{f} : Q(C, R) \rightarrow C'$  with  $f = \Delta(\hat{f}) \circ \pi$ , in other words, quotient map  $\pi : C \rightarrow Q(C, R)$  is a common coequalizer of all equivalent arrows  $f \sim_R g$  with arbitrary domain and codomain  $C$  and, in particular, is always an epimorphism).

Quotient functor may not exist for the whole category  $\mathbf{C}_{\mathcal{R}}$ , but there always exists a (maximal)

full subcategory  $\mathbf{C}_{\mathcal{R}Q} \hookrightarrow \mathbf{C}_{\mathcal{R}}$  for which  $\mathbf{C}_{\mathcal{R}Q} \begin{array}{c} \xrightarrow{Q} \\ \perp \\ \xleftarrow{\Delta} \end{array} \mathbf{C}$  (indeed,  $\mathbf{C}_{\mathcal{R}Q}$  is always non empty since  $Q \circ \Delta(C) = C$ , i.e.  $\Delta(C) \in \text{Ob}(\mathbf{C}_{\mathcal{R}Q})$ ).

If  $\mathbf{C}$  is a concrete category with representable underlying functor  $U := \mathbf{C}(I, -)$  then to each equivalence relation  $R : \mathbf{C}^{op} \rightarrow \mathcal{R}$  on object  $C$  with quotient map  $\pi : (C, R) \rightarrow \Delta(Q(C, R))$  there corresponds a usual equivalence relation on  $\mathbf{C}(I, C)$  with quotient map  $\pi \circ - : \mathbf{C}(I, C) \rightarrow \mathbf{C}(I, Q(C, R))$ , and, conversely, to usual equivalence relation on  $\mathbf{C}(I, C)$  with quotient map  $\pi \circ - : \mathbf{C}(I, C) \rightarrow \mathbf{C}(I, C')$  there corresponds a maximal 'saturated' equivalence relation  $R : \mathbf{C}^{op} \rightarrow \mathcal{R}$  on object  $C$  with quotient map  $\pi : C \rightarrow C' \equiv Q(C, R)$  such that  $f \sim_R g$  iff  $\pi \circ f = \pi \circ g$ . In general, equivalence relation on hom-sets is weaker than usual one.

Let  $C \in \text{Ob } \mathbf{C}$ ,  $\sigma : G \rightarrow \mathbf{Aut}_{\mathbf{C}}(C)$ , then  $G$  also acts on hom-sets  $\mathbf{C}(C', C)$ ,  $C' \in \text{Ob } \mathbf{C}$ ,  $G \times \mathbf{C}(C', C) \rightarrow \mathbf{C}(C', C) : \begin{cases} (g, f) \mapsto \sigma(g) \circ f & \text{left action} \\ (g, f) \mapsto \sigma(g^{-1}) \circ f & \text{right action} \end{cases}$ , i.e.  $\exists$  a functor  $\Sigma : \mathbf{C}^{op} \rightarrow$

$$\begin{array}{ccc} \mathbf{C}^{op} & \xrightarrow{\Sigma} & G\text{-}\mathbf{Set} \\ & \searrow \mathbf{C}(-, C) & \downarrow p \\ & & \mathbf{Set} \end{array} \quad \text{(it means that all hom-sets } \mathbf{C}(C', C), C' \in \text{Ob } \mathbf{C}, \text{ are regarded with the given } G\text{-action).}$$

There are functors

- $r : G\text{-}\mathbf{Set} \rightarrow \mathcal{R} : \begin{cases} (X, G, \sigma) \mapsto (X, R_{\sigma}) & \text{on objects} \\ ((X, G, \sigma) \xrightarrow{f} (X', G, \sigma')) \mapsto ((X, R_{\sigma}) \xrightarrow{f} (X', R_{\sigma'})) & \text{on arrows} \end{cases}$   
(where  $R_{\sigma}$  is an equivalence relation on  $X$  such that  $(x, y) \in R_{\sigma}$  iff  $\exists g \in G \ y = \sigma(g)x$ )

$r$  is a functor over **Set**, i.e.

$$\begin{array}{ccc} G\text{-}\mathbf{Set} & \xrightarrow{r} & \mathcal{R} \\ & \searrow p & \downarrow p \\ & & \mathbf{Set} \end{array}$$

•  $r : G\text{-}\mathbf{C} \rightarrow \mathbf{C}_{\mathcal{R}} : \begin{cases} (C, G, \sigma) \mapsto (C, R_{\sigma}) & \text{on objects} \\ ((C, G, \sigma) \xrightarrow{f} (C', G, \sigma')) \mapsto ((C, R_{\sigma}) \xrightarrow{f} (C', R_{\sigma'})) & \text{on arrows} \end{cases}$

(where  $R_{\sigma} := r \circ \Sigma$  is an equivalence relation on object  $C$  corresponding to  $\sigma$ ,

$$\begin{array}{ccc} \mathbf{C}^{op} & \xrightarrow{R_{\sigma}} & \mathcal{R} \\ & \searrow \mathbf{C}(-, C) & \downarrow p \\ & & \mathbf{Set} \end{array})$$

$r$  is a functor over  $\mathbf{C}$ , i.e.

$$\begin{array}{ccc} G\text{-}\mathbf{C} & \xrightarrow{r} & \mathbf{C}_{\mathcal{R}} \\ & \searrow p & \downarrow p \\ & & \mathbf{C} \end{array}$$

Let  $G\text{-}\mathbf{C}_Q := r^{-1}(\mathbf{C}_{\mathcal{R}Q})$ . Then  $\exists$  a quotient functor  $G\text{-}\mathbf{C}_Q \xrightarrow{r} \mathbf{C}_{\mathcal{R}Q} \xrightarrow{Q} \mathbf{C}$ . Denote it again by  $Q$ , and  $Q \circ r(C, G, \sigma)$  by  $C/G$ .

For arbitrary functor  $F : \mathbf{C} \rightarrow \mathbf{D}$  we have  $G\text{-}F : G\text{-}\mathbf{C} \rightarrow G\text{-}\mathbf{D}$  such that

$$\begin{array}{ccc} \mathbf{C} & \xrightarrow{F} & \mathbf{D} \\ p \uparrow & & \uparrow p \\ G\text{-}\mathbf{C} & \xrightarrow{G\text{-}F} & G\text{-}\mathbf{D} \end{array} \text{ but } F$$

needs not preserve quotients, i.e. the diagram

$$\begin{array}{ccc} G\text{-}\mathbf{C}_Q & \xrightarrow{G\text{-}F} & G\text{-}\mathbf{D}_Q \\ Q \downarrow & \cong & \downarrow Q \\ \mathbf{C} & \xrightarrow{F} & \mathbf{D} \end{array} \text{ can be wrong (the dotted}$$

arrow may not exist and the natural isomorphism may not hold). If the above diagram holds (up to iso) then  $F : \mathbf{C} \rightarrow \mathbf{D}$  **preserves quotients** (of category  $G\text{-}\mathbf{C}$ ). In this case  $F(C/G) \cong$

$F(C)/G$ . Quotient  $C/G$  is called **universal** [Kom] if  $\forall C' \in \text{Ob } \mathbf{C}$

$$\begin{array}{ccc} C' \times C & & \\ \pi \downarrow & \searrow 1 \times p & \\ (C' \times C)/G & \xrightarrow{\sim} & C' \times (C/G) \end{array}$$

( $C'$  is with trivial  $G$ -action).

**Proposition 8.1.** Let  $\begin{array}{c} \mathbf{E} \\ p \downarrow \\ \mathbf{B} \end{array}$  be a structure on  $\mathbf{B}$ ,  $p$  preserve quotients of category  $G\text{-}\mathbf{E}$ , and

$(E, G, \sigma) \in \text{Ob}(G\text{-}\mathbf{E})$  be an object such that  $E/G$  exists with  $\pi : E \rightarrow E/G$ , canonical projection, then  $\begin{pmatrix} E/G \\ p(E/G) \end{pmatrix} = (p(\pi))_* \begin{pmatrix} E \\ p(E) \end{pmatrix}$  is a direct image of  $\begin{pmatrix} E \\ p(E) \end{pmatrix}$ .

*Proof.* We need to prove that  $\begin{pmatrix} E \\ p(E) \end{pmatrix} \xrightarrow{\begin{pmatrix} \pi \\ p(\pi) \end{pmatrix}} \begin{pmatrix} E/G \\ p(E/G) \end{pmatrix}$  is cocartesian. Take  $\begin{pmatrix} u \\ v \end{pmatrix} : \begin{pmatrix} E \\ p(E) \end{pmatrix} \rightarrow \begin{pmatrix} E' \\ p(E') \end{pmatrix}$  such that  $v = k \circ p(\pi)$  for some  $k : p(E/G) \rightarrow p(E')$ , i.e.  $\forall f \sim_G f' : B \rightarrow p(E) \quad v \circ f =$

$v \circ f'$ .

Assume,  $h \sim_G h' : E_1 \rightarrow E$  then  $p(h) \sim_G p(h') : p(E_1) \rightarrow p(E)$  (because, if  $h' = \sigma(g) \circ h$  then  $p(h') = p(\sigma(g)) \circ p(h)$ ). So,  $v \circ p(h) = v \circ p(h')$ ,  $p(u) \circ p(h) = p(u) \circ p(h')$ ,  $u \circ h = u \circ h'$  ( $p$  is faithful), i.e.  $u$  coequalizes all  $\sim_G$ -equivalent arrows (in  $R_\sigma$ ).

Therefore,  $u = \hat{u} \circ \pi$  for a unique  $\hat{u} : E/G \rightarrow E'$ .

$(v = p(u) = p(\hat{u}) \circ p(\pi) = k \circ p(\pi)) \Rightarrow (p(\hat{u}) = k)$  by universality of  $p(\pi)$ .

Finally,  $\exists! \begin{pmatrix} \hat{u} \\ k \end{pmatrix} : \begin{pmatrix} E/G \\ p(E/G) \end{pmatrix} \rightarrow \begin{pmatrix} E' \\ p(E') \end{pmatrix}$  such that  $\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \hat{u} \\ k \end{pmatrix} \circ \begin{pmatrix} \pi \\ p(\pi) \end{pmatrix}$ , i.e.  $\begin{pmatrix} \pi \\ p(\pi) \end{pmatrix}$  is cocartesian.  $\square$

### 8.1. Group objects, subgroups, quotient objects.

**Definition 8.1.1.** Let  $\mathbf{C}$  be a category with binary products and terminal object 1.

- $G \in \text{Ob } \mathbf{C}$  is called a **group object** if  $\exists$  maps  $m : G \times G \rightarrow G$ ,  $e : 1 \rightarrow G$ ,  $inv : G \rightarrow G$  such that the following group-like diagrams hold

$$\begin{array}{ccccc} G \times G \times G & \xrightarrow{1 \times m} & G \times G & & G \times G \xrightarrow{1 \times inv} G \times G \xleftarrow{inv \times 1} G \times G \\ m \times 1 \downarrow & & \downarrow m & & \uparrow \Delta \quad \downarrow m \quad \uparrow \Delta \\ G \times G & \xrightarrow{m} & G & & G \xrightarrow{e \circ !} G \xleftarrow{e \circ !} G \end{array}$$

$\begin{array}{ccc} G \times 1 & \xrightarrow{1 \times e} & G \times G \xleftarrow{e \times 1} 1 \times G \\ & \searrow p_1 & \downarrow m \swarrow p_2 \\ & & G \end{array}$

- Subobject  $K \rightarrowtail G$  of group object  $G$  is called a **subgroup** (object) if  $\exists$  maps  $m_K :$

$$\begin{array}{c} K \times K \rightarrowtail K, \quad e_K : 1 \rightarrow K, \quad inv_K : K \rightarrow K \text{ such that } K \times K \xrightarrow{m_K} G \times G \xrightarrow{m} G \leftarrowtail K \\ \\ 1 \xrightarrow{e_K} K \xrightarrow{e} G \quad K \xrightarrow{inv_K} G \xrightarrow{inv} G \leftarrowtail K \end{array}$$

- For two elements  $f, g : 1 \rightarrow G$  **multiplication**  $f \cdot g : 1 \rightarrow G$  is  $1 \xrightarrow{\langle f, g \rangle} G \times G \xrightarrow{m} G$

- **Right shift**  $R_g : G \rightarrow G$  (by element  $g : 1 \rightarrow G$ ) is  $G \xrightarrow{\sim} G \times 1 \xrightarrow{1 \times g} G \times G \xrightarrow{m} G$

$$\begin{array}{c} \text{Left shift } L_g : G \rightarrow G \text{ (by element } g : 1 \rightarrow G) \text{ is } G \xrightarrow{\sim} 1 \times G \xrightarrow{g \times 1} G \times G \xrightarrow{m} G \quad \square \\ \\ \end{array}$$

**Proposition 8.1.1.** For a group object  $G \in \text{Ob } \mathbf{C}$

- $\mathbf{C}(1, G)$  is a group,
- $\exists$  (anti)representation  $\mathbf{C}(1, G) \rightarrow \mathbf{Aut}_{\mathbf{C}}(G) : g \mapsto R_g$  (by right shifts) and representation  $\mathbf{C}(1, G) \rightarrow \mathbf{Aut}_{\mathbf{C}}(G) : g \mapsto L_g$  (by left shifts).

*Proof.*

- It follows immediately from group object axioms ( $e : 1 \rightarrow G$  is the identity,  $inv \circ g : 1 \rightarrow G$  is the inverse of  $g : 1 \rightarrow G$ ).
- $R_e = 1_G : G \rightarrow G$  (obvious)
- $R_f \circ R_g = R_{g \circ f}$  follows from the diagram

$$\begin{array}{ccccccc}
G & \xrightarrow{\sim} & G \times 1 & \xrightarrow{1 \times g} & G \times G & \xrightarrow{m} & G \xrightarrow{\sim} G \times 1 \xrightarrow{1 \times f} G \times G \xrightarrow{m} G \\
& \searrow \sim & & & & & \nearrow m \\
& & G \times 1 & \xrightarrow{1 \times (g \cdot f)} & G \times G & & \\
& & \nearrow 1 \times \langle 1, 1 \rangle & \searrow 1 \times \langle g, f \rangle & \nearrow 1 \times m & & \\
G \times 1 \times 1 & \xrightarrow{1 \times g \times f} & G \times G \times G & \xrightarrow{m \times 1} & G \times G & & \\
& \searrow (m \circ (1 \times g)) \times 1 & & & \nearrow 1 \times f & & \\
& & G \times 1 & & & & 
\end{array}$$

Two dotted paths  $G \times 1 \xrightarrow{1 \times g} G \times G \xrightarrow{m} G \xrightarrow{p_1^{-1}} G \times 1$  and

$G \times 1 \xrightarrow{1 \times \langle 1, 1 \rangle} G \times 1 \times 1 \xrightarrow{(m \circ (1 \times g)) \times 1} G \times 1$  are equal since their composites with projections  $p_1 : G \times 1 \xrightarrow{\sim} G$  and  $p_2 \equiv ! : G \times 1 \rightarrow 1$  are equal, indeed,  $\begin{cases} p_1 \circ (p_1^{-1} \circ m \circ (1 \times g)) = m \circ (1 \times g) \\ p_2 \circ (p_1^{-1} \circ m \circ (1 \times g)) = ! \end{cases}$  and  $\begin{cases} p_1 \circ (((m \circ (1 \times g)) \times 1) \circ (1 \times \langle 1, 1 \rangle)) = p_1 \circ (((m \circ (1 \times g)) \times 1) \circ (\langle 1, ! \rangle \times 1)) = * \\ p_2 \circ (((m \circ (1 \times g)) \times 1) \circ (1 \times \langle 1, 1 \rangle)) = ! \end{cases}$  and  $\begin{cases} * = p_1 \circ ((m \circ (1 \times g)) \circ \langle 1, ! \rangle \times 1) = m \circ (1 \times g) \circ \langle 1, ! \rangle \circ p_1 = m \circ (1 \times g) \circ 1_{G \times 1} = * \\ * = m \circ (1 \times g) \end{cases}$

Proof for left shift  $L_g$  is similar.  $\square$

**Corollary.** If  $K \rightharpoonup G$  is a subgroup (object) of  $G$  then  $\mathbf{C}(1, K) \xrightarrow{\sigma} \mathbf{Aut}_{\mathbf{C}}(G) : k \mapsto R_k$  is a (right) action  $(G, K, \sigma)$  on  $G$  (by right shifts from  $K$ ). If quotient  $Q(G, K, \sigma) \in \mathbf{Ob} \mathbf{C}$  exists it is called **quotient object**  $G/K$  (under right action of  $K$ ).  $\square$

**Proposition 8.1.2.** Let  $p : G \rightarrow G/K$  be a quotient map s.t.  $\mathbf{C}(1, p) : \mathbf{C}(1, G) \rightarrow \mathbf{C}(1, G/K)$  is surjective. Then

•  $L_g : G \rightarrow G$  induces a **Set-map**  $\bar{L}'_g : \mathbf{C}(1, G/K) \xrightarrow{\sim} \mathbf{C}(1, G/K)$

• If  $G/K$  is universal then  $\exists \bar{L}_g : G/K \xrightarrow{\sim} G/K$  in  $\mathbf{C}$  s.t.  $\mathbf{C}(1, \bar{L}_g) = \bar{L}'_g$  and

$$\begin{array}{ccc}
G & \xrightarrow{L_g} & G \\
p \downarrow & & \downarrow p \\
G/K & \xrightarrow{\bar{L}_g} & G/K
\end{array}$$

*Proof.*

• Claim 1:  $\bar{L}'_g : px \mapsto pL_gx$  is well-defined and iso

$$\begin{array}{ccc}
G & \xrightarrow{L_g} & G \\
\downarrow p & \swarrow x & \searrow L_gx \\
& 1 & \\
\downarrow p & \swarrow px & \searrow pL_gx \\
G/K & & G/K
\end{array}$$

Proof of Claim 1: If  $px = px'$  then  $\exists k \in \mathbf{C}(1, K)$  such that  $x' = R_kx = x \cdot k$ . Then  $L_gx' = L_g(x \cdot k) = g \cdot (x \cdot k) = (g \cdot x) \cdot k = (L_gx) \cdot k = R_k(L_gx)$ , i.e.  $pL_gx' = pL_gx$ .

$\bar{L}'_{g^{-1}} \circ \bar{L}'_g(px) = \bar{L}'_{g^{-1}}(pL_gx) = p(L_{g^{-1}}L_gx) = px$ .  $\square$

• Claim 2:  $G \times G \xrightarrow{1 \times p} G \times G/K$  is a quotient map of  $(G \times G, K, \langle 1, \sigma \rangle) \in \mathbf{Ob} \mathbf{C}\text{-}K$ , where  $\mathbf{C}\text{-}K$  is a category of right actions of  $\mathbf{C}(1, K)$  on objects of  $\mathbf{C}$ ,  $\langle 1, \sigma \rangle : \mathbf{C}(1, K) \rightarrow \mathbf{Aut}_{\mathbf{C}}(G \times G) : k \mapsto 1 \times R_k$ .

Proof of Claim 2 follows immediately from the definition of universal quotient.  $\square$

Claim 3:  $\forall k : 1 \rightarrow G \quad m \circ (1 \times R_k) = R_k \circ m$ .

Proof of Claim 3 follows from the diagram

$$\begin{array}{ccccccc}
 & & & 1 \times R_k & & & \\
 G \times G & \xrightarrow{\sim} & G \times G \times 1 & \xrightarrow{1 \times 1 \times k} & G \times G \times G & \xrightarrow{1 \times m} & G \times G \xrightarrow{m} G \\
 \downarrow m & & \downarrow m \times 1 & & \downarrow m \times 1 & & \nearrow m \\
 G & \xrightarrow{\sim} & G \times 1 & \xrightarrow{1 \times k} & G \times G & & \\
 & & & R_k & & & 
 \end{array}$$

The dotted paths  $p_1^{-1} \circ m, (m \times 1) \circ (1 \times p_1^{-1}) : G \times G \rightarrow G \times 1$  are equal since their composites with projections  $p_1 : G \times G \rightarrow G, p_2 = ! : G \times G \rightarrow 1$  are equal:  $\begin{cases} p_1 \circ (p_1^{-1} \circ m) = m \\ p_2 \circ (p_1^{-1} \circ m) = ! \end{cases}$  and  $\begin{cases} p_1 \circ (m \times 1) \circ (1 \times p_1^{-1}) = m \times p_{G \times G} \circ (1 \times p_1^{-1}) = m \circ 1_{G \times G} = m \\ p_2 \circ (m \times 1) \circ (1 \times p_1^{-1}) = ! \end{cases}$

[where  $p_{G \times G} : G \times G \times 1 \rightarrow G \times G$  is the projection,  $p_{G \times G} \circ (1 \times p_1^{-1}) = 1_{G \times G}$  since  $p_{G \times G} \circ (1 \times p_1^{-1}) \circ \langle x, y \rangle = p_{G \times G} \circ \langle x, y, ! \rangle = \langle x, y \rangle$ ].  $\square$

$$\begin{array}{ccc}
 G \times G & \xrightarrow{m} & G \\
 1 \times p \downarrow & & \downarrow p \\
 G \times G/K & \xrightarrow{\exists ! \bar{m}} & G/K
 \end{array}$$

Proof of Claim 4: Take  $\langle f, g \rangle \sim_K \langle f', g' \rangle : X \rightarrow G \times G$  (where  $G \times G$  is with the action  $\mathbf{C}(1, K) \ni k \mapsto 1 \times R_k \in \mathbf{Aut}_{\mathbf{C}}(G \times G)$ ) then  $\langle f', g' \rangle = (1 \times R_k) \circ \langle f, g \rangle$  for some  $k \in \mathbf{C}(1, K)$ .

$p \circ m \circ \langle f', g' \rangle = p \circ m \circ (1 \times R_k) \circ \langle f, g \rangle = p \circ R_k \circ m \circ \langle f, g \rangle = p \circ m \circ \langle f, g \rangle$  [ $p \circ R_k = p$  by definition of quotient  $p$ ]. So,  $p \circ m$  coequalizes  $\sim_K$ -equivalent arrows to  $G \times G$ . Therefore,  $\exists ! \bar{m} : G \times G/K \rightarrow G/K$  filling out the above diagram.  $\square$

Now, define  $\bar{L}_g : G/K \rightarrow G/K$  as  $G/K \xrightarrow{\sim} 1 \times G/K \xrightarrow{g \times 1} G \times G/K \xrightarrow{\bar{m}} G/K$ . It works since

$$\begin{array}{ccccccc}
 & & & L_g & & & \\
 G & \xrightarrow{\sim} & 1 \times G & \xrightarrow{g \times 1} & G \times G & \xrightarrow{m} & G \\
 \downarrow p & & \downarrow 1 \times p & & \downarrow 1 \times p & & \downarrow p \\
 G/K & \xrightarrow{\sim} & 1 \times G/K & \xrightarrow{g \times 1} & G \times G/K & \xrightarrow{\bar{m}} & G/K \\
 & & & \bar{L}_g & & & 
 \end{array}$$

and  $\mathbf{C}(1, \bar{L}_g)(px) = \bar{L}_g \circ p \circ x = p \circ L_g \circ x = p L_g x = \bar{L}_g'(px)$ , i.e.  $\mathbf{C}(1, \bar{L}_g) = \bar{L}_g'$ .  $\square$

## 8.2. C-group actions.

### Definition 8.2.1.

- Let  $G$  be a group object in  $\mathbf{C}$ ,  $X \in \text{Ob } \mathbf{C}$ , then  $\mathbf{C}$ -map  $\rho : G \times X \rightarrow X$  is a (left) **group**

action on  $X$  if

$$\begin{array}{ccc}
 G \times G \times X & \xrightarrow{1 \times \rho} & G \times X \\
 m \times 1 \downarrow & & \downarrow \rho \\
 G \times X & \xrightarrow{\rho} & X
 \end{array}
 \quad
 \begin{array}{ccc}
 1 \times X & \xrightarrow{e \times 1} & G \times X \\
 & \searrow p_2 & \downarrow \rho \\
 & & X
 \end{array}$$

• **Left shift**  $L_g^X : X \rightarrow X$  (by  $g \in \mathbf{C}(1, G)$ ) is the composite

$$\begin{array}{ccc}
 X & \xrightarrow[p_2^{-1}]{\sim} & 1 \times X \xrightarrow{g \times 1} G \times X \\
 & \searrow L_g^X & \downarrow \rho \\
 & & X
 \end{array}$$

- If  $K \xrightarrow{i_1} G$  is a subgroup of  $G$ ,  $Y \xrightarrow{i_2} X$  is a subobject of  $X$  then  $K$  **stabilizes**  $Y$  if

$\exists f : K \times Y \rightarrow Y$  such that

$$\begin{array}{ccc}
 G \times X & \xrightarrow{\rho} & X \\
 i_1 \times i_2 \uparrow & & \uparrow i_2 \\
 K \times Y & \xrightarrow{f} & Y
 \end{array}$$

□

**Lemma 8.2.1.** Let  $Y \xrightarrow{i} X$  be a subobject of object  $X$  with  $G$ -action  $\rho : G \times X \rightarrow X$ . Assignment  $\mathbf{Stab}_Y : \mathbf{Ob} \mathbf{C} \rightarrow \mathbf{Ob} \mathbf{Set} : Z \mapsto \mathbf{Stab}_Y(Z) \subset \mathbf{C}(Z, G)$  such that  $(x : Z \rightarrow G) \in$

$\mathbf{Stab}_Y(Z)$  iff  $\exists \rho_x : Z \times Y \rightarrow Y$  such that

$$\begin{array}{ccc}
 G \times X & \xrightarrow{\rho} & X \\
 1 \times i \uparrow & & \uparrow i \\
 G \times Y & & Y \\
 x \times 1 \uparrow & \nearrow \exists \rho_x & \\
 Z \times Y & & 
 \end{array}$$

is functorial (hom-subfunctor).

*Proof.* For  $(f : W \rightarrow Z) \in \mathbf{Ar} \mathbf{C}$  define  $\mathbf{Stab}_Y(f) : \mathbf{Stab}_Y(Z) \rightarrow \mathbf{Stab}_Y(W) : x \mapsto x \circ f$  (as precomposite with  $f$ ). This is correct since if  $x \in \mathbf{Stab}_Y(Z)$  then  $x \circ f \in \mathbf{Stab}_Y(W)$  which can

be seen from the diagram

$$\begin{array}{ccc}
 G \times X & \xrightarrow{\rho} & X \\
 x \times i \uparrow & & \uparrow i \\
 Z \times Y & \xrightarrow{\rho_x} & Y \\
 f \times 1 \uparrow & \nearrow \rho_{x \circ f} & \\
 W \times Y & & 
 \end{array}$$

Functorial properties of  $\mathbf{Stab}_Y$  are obvious. So,

$\exists$  functor  $\mathbf{Stab}_Y : \mathbf{C}^{op} \rightarrow \mathbf{Set}$  and  $\mathbf{Stab}_Y \hookrightarrow \mathbf{C}(-, G)$  is a hom-subfunctor. □

**Definition 8.2.2.** If  $\mathbf{Stab}_Y : \mathbf{C}^{op} \rightarrow \mathbf{Set}$  is representable then denote its representing object by  $\mathbf{Stab}_Y \in \mathbf{Ob} \mathbf{C}$  and call it a **stabilizer** of  $Y \xrightarrow{\quad} X$  (for group  $G$  acting on  $X$ ). □

**Proposition 8.2.1.** Let  $\mathbf{Stab}_Y : \mathbf{C}^{op} \rightarrow \mathbf{Set}$  be represented by  $\mathbf{Stab}_Y \in \mathbf{Ob} \mathbf{C}$ . Then

- $\mathbf{Stab}_Y \xrightarrow{j} G$  is a subobject of group  $G$  (but not necessarily a group object itself),
- $j$  is the universal element of functor  $\mathbf{Stab}_Y$ ,
- each element in  $\mathbf{Stab}_Y(Z)$  has form  $j \circ x$  for a unique  $x : Z \rightarrow \mathbf{Stab}_Y$ , and all elements of this form  $(\forall x : Z \rightarrow \mathbf{Stab}_Y)$  are in  $\mathbf{Stab}_Y(Z)$  [in other words,  $(z \in_Z G) \& (z \in \mathbf{Stab}_Y(Z)) \Leftrightarrow (z \in_Z \mathbf{Stab}_Y)$ ],

•  $\exists! \rho_j : \mathbf{Stab}_Y \times Y \rightarrow Y$  such that

$$\begin{array}{ccc} G \times X & \xrightarrow{\rho} & X \\ j \times i \uparrow & & \uparrow i \\ \mathbf{Stab}_Y \times Y & \xrightarrow{\rho_j} & Y \end{array}$$

and  $\rho_j$  is universal among arrows with

similar property, i.e.  $\forall \rho_{x'} : Z \times Y \rightarrow Y$  such that

$$\begin{array}{ccc} G \times X & \xrightarrow{\rho} & X \\ x' \times i \uparrow & & \uparrow i \\ Z \times Y & \xrightarrow{\rho_{x'}} & Y \end{array}$$

$\exists! x : Z \rightarrow \mathbf{Stab}_Y$  such

that  $x' = j \circ x$  and

$$\begin{array}{ccc} \mathbf{Stab}_Y \times Y & \xrightarrow{\rho_j} & Y \\ x \times 1 \uparrow & \nearrow \rho_{x'} & \\ Z \times Y & & \end{array}$$

*Proof.* First three points follow from Yoneda Lemma ( $j$  is the universal element of representation  $\mathbf{C}(-, \mathbf{Stab}_Y) \xrightarrow{\sim} \mathbf{Stab}_Y$  corresponding (under Yoneda embedding) to monic (natural transformation)  $\mathbf{C}(-, \mathbf{Stab}_Y) \xrightarrow{\sim} \mathbf{Stab}_Y \hookrightarrow \mathbf{C}(-, G)$ ). Fourth point follows from the definition and above properties of functor  $\mathbf{Stab}_Y$  and that  $Y \xrightarrow{i} X$  is monic.  $\square$

**Lemma 8.2.2.** *Subobject  $H \hookrightarrow G$  of a group object  $G \in \mathbf{Ob} \mathbf{C}$  is itself a group object iff  $\forall Z \in \mathbf{Ob} \mathbf{C}$  there are induced group operations in hom-set  $\mathbf{C}(Z, H)$  in the following way*

$$\begin{array}{ccc} \begin{array}{ccc} G \times G & \xrightarrow{m} & G \\ \uparrow & & \uparrow \\ H \times H & & H \\ \uparrow \langle x, y \rangle & \nearrow \exists m(\langle x, y \rangle) & \\ Z & & \end{array} & \begin{array}{ccc} G & \xrightarrow{inv} & G \\ \uparrow & & \uparrow \\ H & & H \\ \uparrow x & \nearrow \exists inv(x) & \\ Z & & \end{array} & \begin{array}{ccc} 1 & \xrightarrow{e} & G \\ \uparrow ! & & \uparrow \\ Z & \xrightarrow{\exists e(!)} & H \end{array} \end{array}$$

*Proof* is obvious in both directions.  $\square$

**Proposition 8.2.2.**

- $\mathbf{Stab}_Y \hookrightarrow G$  is always a submonoid of group object  $G \in \mathbf{Ob} \mathbf{C}$ .
- $\mathbf{Stab}_Y \hookrightarrow G$  is a subgroup of group object  $G \in \mathbf{Ob} \mathbf{C}$  if  $\forall Z \in \mathbf{Ob} \mathbf{C} \forall x \in \mathbf{Stab}_Y(Z)$  the

corresponding map  $\rho_x : Z \times Y \rightarrow Y$  as in the diagram

$$\begin{array}{ccc} G \times X & \xrightarrow{\rho} & X \\ x \times i \uparrow & & \uparrow i \\ Z \times Y & \xrightarrow{\rho_x} & Y \end{array}$$

is surjective in

the second argument, i.e.  $\forall t : T \rightarrow Z$  the map  $\mathbf{C}(T, Y) \ni s \mapsto \rho_{x \circ t} \in \mathbf{C}(T, Y)$  is surjective [it holds in classical case in  $\mathbf{Set}$ ].

*Proof.*



•

$$\begin{array}{ccc}
& & G \times X \\
& \nearrow m \times 1 & \searrow \rho \\
G \times G \times X & \xrightarrow{1 \times \rho} & G \times X \xrightarrow{\rho} X \\
\uparrow x \times y \times i & & \uparrow x \times i \quad \uparrow i \\
Z \times Z \times Y & \xrightarrow{1 \times \rho_y} & Z \times Y \xrightarrow{\rho_x} Y \\
\uparrow \langle 1, 1 \rangle \times 1 & & \nearrow \rho_{m(\langle x, y \rangle)} \\
Z \times Y & &
\end{array}$$

i.e.  $m(\langle x, y \rangle) \in \mathbf{Stab}_Y(Z)$  if  $x, y \in \mathbf{Stab}_Y(Z)$   
(the same,  $m(\langle x, y \rangle) \in {}_Z \mathbf{Stab}_Y$  if  $x, y \in {}_Z \mathbf{Stab}_Y$ )

$$\begin{array}{ccc}
G \times X & \xrightarrow{\rho} & X \\
\uparrow e \times 1 & & \uparrow 1 \\
1 \times X & \xrightarrow{p_2} & X \\
\uparrow 1 \times i & & \uparrow i \\
1 \times Y & \xrightarrow{p_2} & Y \\
\uparrow ! \times 1 & & \nearrow p_2 = \rho_{e(!)} \\
Z \times Y & &
\end{array}$$

i.e.  $e(!) := e \circ ! \in \mathbf{Stab}_Y(Z)$  (the same,  $e(!) \in {}_Z \mathbf{Stab}_Y$ )

• In general, for  $x \in {}_Z \mathbf{Stab}_Y$   $\text{inv}(x) \in {}_Z G$ , but  $\text{inv}(x) \notin {}_Z \mathbf{Stab}_Y$ .

$$\begin{array}{ccc}
& & G \times X \\
& \nearrow m \times 1 & \searrow \rho \\
G \times G \times X & \xrightarrow{1 \times \rho} & G \times X \xrightarrow{\rho} X \\
\uparrow (e \circ !) \times i & & \uparrow \text{inv}(x) \times x \times i \quad \uparrow \text{inv}(x) \times i \quad \uparrow i \\
Z \times Z \times Y & \xrightarrow{1 \times \rho_x} & Z \times Y \xrightarrow{\rho_{\text{inv}(x)}} Y \\
\uparrow \langle 1, 1 \rangle \times 1 & & \nearrow p_2 = \rho_{e(!)} \\
Z \times Y & & \\
\uparrow \langle t, s \rangle & & \\
T & &
\end{array}$$

Why does  $\rho_{\text{inv}(x)}$  exist?

**Lemma 8.2.3.** If a square

$$\begin{array}{ccc}
\text{Hom}(-, A) & \xrightarrow{f} & \text{Hom}(-, B) \\
h \downarrow & \swarrow \exists d & \downarrow g \\
\text{Hom}(-, C) & \xrightarrow{k} & \text{Hom}(-, D)
\end{array}$$

of natural transformations of representables commutes, where  $f$  and  $k$  are respectively componentwise surjective and componentwise injective maps, then  $\exists$  a (unique) diagonal  $d$  keeping the diagram commutative.

*Proof of Lemma.*

$$\begin{array}{ccc}
 \text{Hom}(X, A) & \xrightarrow{f_X} & \text{Hom}(X, B) \\
 \downarrow h_X & \nearrow g'_X & \downarrow g_X \\
 & \text{Im}(k) & \\
 \downarrow k'_X & \searrow & \downarrow \\
 \text{Hom}(X, C) & \xrightarrow{k_X} & \text{Hom}(X, D)
 \end{array}$$

(Note:  $k'_X$  is labeled with a tilde  $\sim$  near the arrow to  $\text{Im}(k)$ )

$k_X, g_X$  factor through  $\text{Im}(k)$  since  $k_X$  is injective and  $f_X$  is surjective. Define diagonal  $d_X := (k'_X)^{-1} \circ g'_X$ . Arrows  $d_X$  ( $X$  is a parameter) form natural transformation which can be seen from the diagram

$$\begin{array}{ccc}
 & \bullet & \bullet \\
 & \swarrow & \searrow \\
 \bullet & \xrightarrow{k} & \bullet \\
 \downarrow d_{X'} & \nearrow & \downarrow d_X \\
 \bullet & \xrightarrow{\text{monic}} & \bullet
 \end{array}$$

□

So, apply the above lemma to the square

$$\begin{array}{ccc}
 \mathbf{C}(-, Z \times Y) & \xrightarrow{(1 \times \rho_x) \circ (<1, 1> \times 1)} & \mathbf{C}(-, Z \times Y) \\
 \downarrow p_2 & \swarrow \exists \rho_{\text{inv}(x)} & \downarrow \rho \circ (\text{inv}(x) \times i) \\
 \mathbf{C}(-, Y) & \xrightarrow{i} & \mathbf{C}(-, X)
 \end{array}$$

(The top arrow is componentwise surjective since  $(1 \times \rho_x) \circ (<1, 1> \times 1) \circ <t, s> = (1 \times \rho_x) \circ <t, t, s> = <t, \rho_x \circ <t, s>>$ , and, so that,  $\forall <m, l>: T \rightarrow Z \times Y \exists$  its preimage  $<t, s>: T \rightarrow Z \times Y$  with  $t = m$  and  $s$  is a solution of the equation  $\rho_x \circ <m, s> = l$  [which exists because  $\rho_x$  is surjective in the second argument]. The bottom arrow is componentwise injective since  $i$  is monic.)

Consequently,  $\exists(!) \rho_{\text{inv}(x)} : Z \times Y \rightarrow Y$  such that

$$\begin{array}{ccc}
 G \times X & \xrightarrow{\rho} & X \\
 \uparrow \text{inv}(x) \times i & & \uparrow i \\
 Z \times Y & \xrightarrow{\rho_{\text{inv}(x)}} & Y
 \end{array}$$

i.e. if  $x \in_Z \mathbf{Stab}_Y$

then  $\text{inv}(x) \in_Z \mathbf{Stab}_Y$ .

□

**Lemma 8.2.4.** *If  $L_g : X \xrightarrow{\sim} X$  is a left shift, and  $Y, Z \rightarrowtail X$  are subobjects of  $X$  such that*

*an induced isomorphism  $\overline{L}_g : Y \xrightarrow{\sim} Z$  exists, i.e. the diagram*

$$\begin{array}{ccc}
 X & \xrightarrow{L_g} & X \\
 \uparrow i_Y & & \uparrow i_Z \\
 Y & \xrightarrow[\exists \overline{L}_g]{\sim} & Z
 \end{array}$$

*commutes, then  $\exists$*

*an induced map (iso)  $\overline{L}_g \circ R_{g^{-1}} : \mathbf{Stab}_Y \xrightarrow{\sim} \mathbf{Stab}_Z$ , corresponding to  $\overline{L}_g : Y \xrightarrow{\sim} Z$ , such that the*

following diagram commutes

$$\begin{array}{ccccc}
 G \times X & \xrightarrow{\rho} & X & & \\
 \uparrow j_Z \times i_Z & \nwarrow (L_g \circ R_{g^{-1}}) \times L_g & \uparrow L_g & & \\
 G \times X & \xrightarrow{\rho} & X & & \\
 \uparrow j_Y \times i_Y & \nwarrow \rho_{j_Z} & \uparrow i_Z & & \\
 \text{Stab}_Z \times Z & \xrightarrow{\rho_{j_Z}} & Z & & \\
 \uparrow \overline{L_g \circ R_{g^{-1}}} \times \overline{L_g} & \nwarrow \sim & \uparrow \overline{L_g} & & \\
 \text{Stab}_Y \times Y & \xrightarrow{\rho_{j_Y}} & Y & & 
 \end{array}$$

*Proof.* The only difficulty is that the left side square commutes, namely,

$$\begin{array}{ccc}
 G & \xleftarrow{L_g \circ R_{g^{-1}}} & G \\
 j_Z \uparrow & & \uparrow j_Y \\
 \text{Stab}_Z & \xleftarrow[\overline{L_g \circ R_{g^{-1}}}]{} & \text{Stab}_Y
 \end{array}$$

Sufficient to show that  $(L_g \circ R_{g^{-1}}) \circ j_Y \in \text{Stab}_Y \text{Stab}_Z$ , i.e. that  $(L_g \circ R_{g^{-1}}) \circ j_Y \in \text{Stab}_Z(\text{Stab}_Y)$ ,

$$\begin{array}{ccc}
 G \times X & \xrightarrow{\rho} & X \\
 \uparrow ((L_g \circ R_{g^{-1}}) \circ j_Y) \times i_Z & & \uparrow i_Z \\
 \text{Stab}_Y \times Z & \xrightarrow[\rho']{} & Z
 \end{array}$$

or that  $\exists \rho' : \text{Stab}_Y \times Z \rightarrow Z$  such that

Indeed,  $((L_g \circ R_{g^{-1}}) \circ j_Y) \times (L_g \circ i_Y) = ((L_g \circ R_{g^{-1}}) \circ j_Y) \times (i_Z \circ \overline{L_g}) = (((L_g \circ R_{g^{-1}}) \circ j_Y) \times i_Z) \circ (1 \times \overline{L_g})$ , then  $\rho \circ (((L_g \circ R_{g^{-1}}) \circ j_Y) \times i_Z) \circ (1 \times \overline{L_g}) = i_Z \circ \overline{L_g} \circ \rho_{j_Y}$ , and so,  $\rho \circ (((L_g \circ R_{g^{-1}}) \circ j_Y) \times i_Z) = i_Z \circ \overline{L_g} \circ \rho_{j_Y} \circ (1 \times \overline{L_g})^{-1}$ , i.e.  $\rho' := \overline{L_g} \circ \rho_{j_Y} \circ (1 \times \overline{L_g})^{-1}$ . Therefore,  $\forall x \in_T \text{Stab}_Y$   $L_g \circ R_{g^{-1}}(x) \in_T \text{Stab}_Z$ , i.e.  $\exists$  the induced map  $\overline{L_g \circ R_{g^{-1}}} : \text{Stab}_Y \rightarrow \text{Stab}_Z$ .  $\square$

### Proposition 8.2.3.

- Two objects  $(G, \mathbf{C}(1, \text{Stab}_Y), \sigma_1)$  and  $(G, \mathbf{C}(1, \text{Stab}_Z), \sigma_2)$  from **C-Grp** (a category of right group actions on objects from **C**) are (equivariantly) isomorphic if  $\exists g \in \mathbf{C}(1, G)$  and an induced isomorphism (as in lemma 2.7.2.4)  $\overline{L_g} : Y \xrightarrow{\sim} Z$ . The required isomorphism has form

$$(G, \mathbf{C}(1, \text{Stab}_Y), \sigma_1) \xrightarrow[\sim]{(L_g \circ R_{g^{-1}}, \overline{L_g \circ R_{g^{-1}} \circ -})} (G, \mathbf{C}(1, \text{Stab}_Z), \sigma_2).$$

- $G/\text{Stab}_Y \simeq G/\text{Stab}_Z$  (if these quotients exist).

*Proof.*

- It is necessary to prove that  $\forall g : 1 \rightarrow G$  and  $k : 1 \rightarrow \text{Stab}_Y$   $L_g \circ R_{g^{-1}} \circ R_k = R_{g \cdot k \cdot g^{-1}} \circ L_g \circ R_{g^{-1}}$ . It follows from two facts  $R_{g \cdot k \cdot g^{-1}} = R_{g^{-1}} \circ R_k \circ R_g$  (antihomomorphism) and commutativity of left and right shifts  $L_{g_1} \circ R_{g_2} = R_{g_2} \circ L_{g_1}$  [the last fact follows from associativity axiom  $\forall < t, r, s > : T \rightarrow G \times G \times G$   $m(t, m(r, s)) = m(m(t, r), s)$ , and so,  $L_{g_1} \circ R_{g_2} \circ t = m(g_1 \circ!, m(t, g_2 \circ!)) = m(m(g_1 \circ!, t), g_2 \circ!) = R_{g_2} \circ L_{g_1} \circ t$ ].
- Isomorphic objects in **C-Grp**<sub>Q</sub> have isomorphic quotients in **C** since  $Q : \mathbf{C-Grp}_Q \rightarrow \mathbf{C}$  is a functor. So,  $Q(G, \mathbf{C}(1, \text{Stab}_Y), \sigma_1) \simeq Q(G, \mathbf{C}(1, \text{Stab}_Z), \sigma_2)$ .  $\square$

**Definition 8.2.3.** An object  $X \in \text{Ob } \mathbf{C}$  with a group action  $\rho : G \times X \rightarrow X$  such that  $\forall x : 1 \rightarrow X$  both  $\text{Stab}_x$  and  $G/\text{Stab}_x$  exist, and  $G/\text{Stab}_x$  is universal, is called **homogenous** if  $\exists$

an isomorphism  $f : G/\text{Stab}_x \xrightarrow{\sim} X$  such that

$$\begin{array}{ccc} G \times (G/\text{Stab}_x) & \xrightarrow{\rho'} & G/\text{Stab}_x \\ 1 \times f \downarrow \sim & & \sim \downarrow f \\ G \times X & \xrightarrow{\rho} & X \end{array} \quad (\text{for an } x : 1 \rightarrow X)$$

where  $\rho'$  is defined from

$$\begin{array}{ccc} G \times G & \xrightarrow{m} & G \\ 1 \times p \downarrow & & \downarrow p \\ G \times (G/\text{Stab}_x) & \xrightarrow{\exists! \rho'} & G/\text{Stab}_x \end{array} \quad (1 \times p \text{ and } p \text{ are quotient maps}). \quad \square$$

**Proposition 8.2.4.** *If  $X$  is a homogenous object (with  $G$ -action  $\rho : G \times X \rightarrow X$ ) and  $\mathbf{C}(1, p) : \mathbf{C}(1, G) \rightarrow \mathbf{C}(1, G/\text{Stab}_x)$  is surjective, where  $G \xrightarrow{p} G/\text{Stab}_x$  is a quotient map, then*

- $\mathbf{C}(1, G)$  acts transitively on  $\mathbf{C}(1, X)$ , i.e.  $\forall x, y : 1 \rightarrow X \exists g : 1 \rightarrow G$  such that  $y = L_g \circ x$ ,
- definition of homogenous object  $X \xrightarrow[f^{-1}]{\sim} G/\text{Stab}_x$  does not depend on the choice of  $x : 1 \rightarrow X$ .

*Proof.*

- $\forall a', b' : 1 \rightarrow G/\text{Stab}_x \exists a, b : 1 \rightarrow G$  s.t.  $pa = a', pb = b'$ , and  $\exists g : 1 \rightarrow G$  s.t.  $b = L_g a$ . By proposition 8.1.2,  $\bar{L}'_g(a') = \bar{L}'_g(pa) = pL_g(a) = pb = b'$  (where  $\bar{L}'_g$  is the induced left shift on  $\mathbf{C}(1, G/\text{Stab}_x)$ ). So,  $\mathbf{C}(1, G)$  acts transitively on set  $\mathbf{C}(1, G/\text{Stab}_x)$ , and consequently, on  $\mathbf{C}(1, X)$ .

- Regard the diagram

$$\begin{array}{ccccc} & & G \times G & \xrightarrow{m} & G \\ & \nearrow (L_g \circ R_{g^{-1}}) \times (L_g \circ R_{g^{-1}}) & & \searrow 1 \times p_y & \downarrow p_y \\ G \times G & \xrightarrow{m} & G & & G \times (G/\text{Stab}_y) \xrightarrow{\dots} G/\text{Stab}_y \\ \downarrow 1 \times p_x & & \downarrow p_x & \nearrow (L_g \circ R_{g^{-1}}) \times \alpha & \downarrow \rho'' \\ G \times (G/\text{Stab}_x) & \xrightarrow{\rho'} & G/\text{Stab}_x & & \\ \downarrow 1 \times f \sim & & \downarrow \sim & \nearrow \sim & \downarrow \sim \\ G \times X & \xrightarrow{\rho} & X & & \end{array}$$

[ $\alpha$  exists as a mediating arrow because  $f \circ p_x$  is a quotient map of  $(G, \mathbf{C}(1, \text{Stab}_x), \sigma_1)$ , and  $p_y \circ (L_g \circ R_{g^{-1}})$  coequalizes  $\sim_{\sigma_1}$ -equivalent arrows ( $L_g \circ R_{g^{-1}}$  is equivariant, and  $p_y$  is a quotient map of  $(G, \mathbf{C}(1, \text{Stab}_y), \sigma_2)$ ), essentially,  $\alpha = Q(L_g \circ R_{g^{-1}})$ . The bottom square commutes because  $(1 \times f) \circ (1 \times p_x)$  is a quotient map, and so, epi]

Therefore,

$$\begin{array}{ccc} & G \times (G/\text{Stab}_y) & \xrightarrow{\rho''} G/\text{Stab}_y \\ & \uparrow (L_g \circ R_{g^{-1}}) \times \alpha \sim & \uparrow \sim \alpha \\ \sim \curvearrowleft 1 \times (\alpha \circ L_{g^{-1}}) & G \times X & \xrightarrow{\rho} X \\ & \uparrow (L_{g^{-1}} \circ R_g) \times L_{g^{-1}} \sim & \uparrow \sim L_{g^{-1}} \\ & G \times X & \xrightarrow{\rho} X \end{array} \quad \text{i.e. } L_g \circ \alpha^{-1} \text{ is the}$$

required isomorphism (by definition 8.2.3).  $\square$

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